ISRAEL JOURNAL OF MATHEMATICS **158** (2007), 249–283
DOI: 10.1007/s11856-007-0013-y

# RELATIVE ENTROPY TUPLES, RELATIVE U.P.E. AND C.P.E. EXTENSIONS

BY

Wen Huang*

*Department of Mathematics, University of Science and Technology of China*
*Hefei, Anhui, 230026, P.R. China*
*and*
*Institut de mathématiques de Luminy*
*163 Avenue de Luminy, case 907, 13288 Marseille cedex 9, France*
*e-mail: wenh@mail.ustc.edu.cn*

AND

Xiangdong Ye** and Guohua Zhang†

*Department of Mathematics, University of Science and Technology of China*
*Hefei, Anhui, 230026, P.R. China*
*e-mail: yexd@ustc.edu.cn, ghzhang@mail.ustc.edu.cn*

ABSTRACT

Relative entropy tuples both in topological and measure-theoretical settings, relative uniformly positive entropy (rel.-u.p.e.) and relative completely positive entropy (rel.-c.p.e.) are studied. It is shown that a relative topological Pinsker factor can be deduced by the smallest closed invariant equivalence relation containing the set of relative entropy pairs. A relative disjointness theorem involving relative topological entropy is proved. Moreover, it is shown that the product of finite rel.-c.p.e. extensions is also rel.-c.p.e..

* The first author is partially supported by NCET, NNSF of China (no. 10401031) and CNRS-K.C.Wong Fellowship.

** The second author is supported by the national key project for basic science (973).

† The third author is supported by NNSF of China (no. 10401031).
Received September 9, 2005

## 1. Introduction

Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretical dynamical system (for short MDS). Then the Pinsker factor of $(X, \mathcal{B}, \mu, T)$ is the maximal zero entropy factor of $(X, \mathcal{B}, \mu, T)$, which is obtained by the smallest sub-$\sigma$-algebra containing all finite partitions with zero entropy. By introducing the notion of entropy pairs, Blanchard and Lacroix [BL] obtained the maximal zero entropy factor for any topological dynamical system (for short TDS), namely the topological Pinsker factor. For a factor map between two TDSs, Glasner and Weiss [GW2] introduced the notion of relative topological Pinsker factor (named $\text{Pinsker}_1$ factor) based on the idea of u.p.e. and c.p.e. extensions. Later on Lemanczyk and Siemaszko [LS] defined another relative topological Pinsker factor (named $\text{Pinsker}_2$ factor) based on the existence of the smallest invariant closed equivalence relation (*icer* for short) with certain properties. In [PS] Park and Siemaszko interpreted the relative topological $\text{Pinsker}_2$ factor using relative measure-theoretical entropy and the discussed relative product. In [HY2] Huang and Ye introduced the notions of entropy tuples both in topological and measure-theoretical settings and established the relationship between the notions.

Recently, Huang, Ye and Zhang [HYZ] obtained some local variational principles for relative entropy, and generalized the corresponding results in [BGH], [LW], [Rm] and [GW4]. As a continuation, in the current paper we shall introduce the notions of relative entropy tuples in both topological and measure-theoretical settings. It turns out that for each $n \geq 2$ the set of relative topological entropy $n$-tuples is just the union of relative measure-theoretical entropy $n$-tuples over all invariant measures, and so the relative topological $\text{Pinsker}_2$ factor is in fact induced by the smallest *icer* containing the set of relative topological entropy pairs. As we think the relative topological $\text{Pinsker}_2$ factor is a more natural notion than the relative topological $\text{Pinsker}_1$ factor, we will refer to it as the relative topological Pinsker factor in the paper.

Based on the notions of relative topological entropy pairs one may define rel.-u.p.e. and rel.-c.p.e. extensions. The properties of rel.-u.p.e. and rel.-c.p.e. extensions are studied. It is known that u.p.e. implies weak mixing, c.p.e. implies the existence of an invariant measure with full support ([B1]), and a u.p.e. system is disjoint from all minimal systems with zero entropy ([B2]). The above properties are obtained in the relative case under some necessary restrictions in the paper. Namely, we show that an open rel.-u.p.e. extension of all orders is disjoint from any relative minimal and relative zero entropy extension; rel.-c.p.e. implies that each fibre either is a singleton or fully supported; and an open rel.-

u.p.e. extension is weakly mixing iff it is weakly mixing of all orders iff the factor is an $E$-system. This answers some questions raised in [PS]. Moreover, we show that the finite product of rel.-u.p.e. extensions (resp. rel.-c.p.e. extensions) has rel.-u.p.e. (resp. rel.-c.p.e.) iff the factors are fully supported. We remark that in [G1] Glasner showed that the finite product of u.p.e. systems is also u.p.e. and our result implies that the finite product of c.p.e. systems is also c.p.e.

This paper is organized as follows. In section 2, the notion of relative topological entropy tuples is introduced and a relative disjointness theorem involving relative topological entropy is obtained. In section 3, the set of relative entropy tuples in a measure-theoretical setting is introduced and characterized. In section 4, the relationship between the set of relative topological entropy tuples and the set of relative measure-theoretical entropy tuples is established, and the relative topological Pinsker factor is interpreted. In section 5, rel.-c.p.e. is studied and characterized and in section 6, rel.-u.p.e. is carefully studied. In the final section, the finite product of rel.-u.p.e. and c.p.e. extensions is investigated. Moreover, in the appendix the relationship between relative topological entropy pairs and asymptotic pairs is discussed.

## 2. Relative topological entropy tuples and relative disjointness

Throughout this paper, by a *topological dynamical system* $(X,T)$ we mean a compact metric space $X$ endowed with a self-homeomorphism $T$. In this section the notion of relative topological entropy tuples is introduced and a well-known disjointness theorem involving entropy is generalized to the relative case. For this purpose, we first need to introduce some notation. For a given a TDS $(X,T)$, a *cover* of $X$ is a finite family of Borel subsets of $X$, whose union is $X$. Denote the set of covers by $\mathcal{C}_X$ and the set of open covers by $\mathcal{C}_X^o$. Given $\mathcal{U},\mathcal{V} \in \mathcal{C}_X$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (write $\mathcal{V} \preceq \mathcal{U}$), if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$.

Let $(X,T)$ and $(Y,S)$ be two TDSs. A continuous map $\pi$: $(X,T) \to (Y,S)$ is called a *homomorphism* or a *factor map* between $(X,T)$ and $(Y,S)$ if it is onto and $\pi T = S\pi$. In this case we say $(X,T)$ is an *extension* of $(Y,S)$ or $(Y,S)$ is a *factor* of $(X,T)$. If in addition $\pi$ is not one-to-one, then it is called *nontrivial.*

Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs and $\mathcal{U} \in \mathcal{C}_X$. For $E \subseteq X$, we define $N(\mathcal{U},E)$ as the minimum among the cardinalities of the subsets of $\mathcal{U}$ which cover $E$, and define $N(\mathcal{U}|\pi) = \sup_{y\in Y} N(\mathcal{U},\pi^{-1}(y))$. Since the sequence $\{\log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}|\pi)\}$ is non-negative and sub-additive, the quantity

$$h_{\rm top}(T,\mathcal{U}|\pi) = \lim_{n\to\infty} \frac{1}{n} \log N\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}|\pi\Big) = \inf_{n\geq 1} \frac{1}{n} \log N\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}|\pi\Big)$$

is well-defined, and is called the *relative topological entropy of* $\mathcal{U}$ *relevant to* $\pi$. The *relative topological entropy relevant to* $\pi$ is defined by

$$h_{\rm top}(T|\pi) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_{\rm top}(T,\mathcal{U}|\pi).$$

When $(Y,S)$ is trivial, we recover the classical topological entropy (see [DGS], [W]), and in this case we shall omit the restriction on $\pi$.

Now, we are going to define the relative topological entropy tuples (r.t.-entropy tuples, for short). Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs and $n \geq 2$. Set $X^{(n)} = X \times \cdots \times X$ ($n$-times), $T^{(n)} = T \times \cdots \times T$ ($n$-times), $\Delta_n(X) = \{(x_i)_1^n \in X^{(n)} | x_1 = \cdots = x_n\}$, the $n$-th diagonal of $X$. We also write $\Delta_X = \Delta_2(X)$.

Let $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$ and $\mathcal{U} \in \mathcal{C}_X$; we say $\mathcal{U}$ is *an admissible cover* with respect to $(x_i)_1^n$, if for any $U \in \mathcal{U}$, $\overline{U} \not\supseteq \{x_1, x_2, \ldots, x_n\}$.

*Definition 2.1:* Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs, and $n \geq 2$. $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$ is called a *r.t.-entropy n-tuple* relevant to $\pi$, if for any admissible cover $\mathcal{U}$ with respect to $(x_i)_1^n$ we have $h_{\rm top}(T,\mathcal{U}|\pi) > 0$.

*Remark 2.2:* It makes no difference if we replace all admissible covers by admissible open covers or admissible closed covers in the above definition. Moreover, we can choose all covers to be of the form $\mathcal{U} = \{U_1, \ldots, U_n\}$, where $U_i^c$ is a neighborhood of $x_i$ such that $U_i^c \cap U_j^c = \emptyset$ when $x_i \neq x_j, 1 \leq i < j \leq n$.

For each $n \geq 2$, denote by $E_n(X,T|\pi)$ the set of all r.t.-entropy $n$-tuples relevant to $\pi$, and write it as $E_n(X,T)$ or $E_n(X)$ when $(Y,S)$ is trivial. Let $R_\pi^{(n)} = \{(x_i)_1^n \in X^{(n)} | \pi(x_1) = \cdots = \pi(x_n)\}$ (denoted by $R_\pi$ when $n = 2$). It is easy to see that $E_n(X,T|\pi) \subseteq R_\pi^{(n)} \setminus \Delta_n(X)$, since for each $(x_i)_1^n \in X^{(n)} \setminus R_\pi^{(n)}$ there is an admissible open cover of the form $\{\pi^{-1}(Y_1), \ldots, \pi^{-1}(Y_m)\}$ which has relative zero entropy relevant to $\pi$, where $Y_1, \ldots, Y_m$ are some open sets of $Y$ with $2 \leq m \leq n$. Following the ideas in [B2], we have

Proposition 2.3: *Let* $\pi_1$: $(X,T) \to (Y,S)$ *be a factor map between TDSs and* $n \geq 2$.

**1:** *Suppose that* $\mathcal{U} = \{U_1, \ldots, U_n\} \in \mathcal{C}_X^o$ *satisfies* $h_{\rm top}(T,\mathcal{U}|\pi_1) > 0$. *Then there exists* $(x_i)_1^n \in E_n(X,T|\pi_1)$ *such that* $x_i \in U_i^c$, $i = 1, \ldots, n$.

**2:** *If* $h_{\mathrm{top}}(T|\pi_1) > 0$*, then* $\overline{E_n(X,T|\pi_1)} \subseteq X^{(n)}$ *is a nonempty invariant closed subset (i.e.* $T^{(n)}(\overline{E_n(X,T|\pi_1)}) = \overline{E_n(X,T|\pi_1)}$*). Moreover,* $\overline{E_n(X,T|\pi_1)} \setminus \Delta_n(X) = E_n(X,T|\pi_1)$*.*

**3:** *Let* $\pi_2\colon (Z,\Theta) \to (X,T)$ *be a factor map between TDSs. Then*

(1) $E_n(X,T|\pi_1) \subseteq \pi_2\times\cdots\times\pi_2(E_n(Z,\Theta|\pi_1\pi_2)) \subseteq E_n(X,T|\pi_1)\cup\Delta_n(X)$.

(2) $E_n(Z,\Theta|\pi_2) \subseteq E_n(Z,\Theta|\pi_1\pi_2)$.

**4:** *Suppose that* $(W,T_W)$ *is a sub-system of* $(X,T)$ *and* $\pi_3\colon W \to \pi_1(W)$ *is the map determined by* $\pi_1$*. Then* $E_n(W,T_W|\pi_3) \subseteq E_n(X,T|\pi_1)$*.*

The notion of *disjointness* of two TDSs was introduced in [F]. Blanchard proved that any u.p.e. system was disjoint from all minimal systems with zero entropy (Proposition 6 in [B2]). In the remaining part of this section, we shall prove a relative version of the result. We start with some notions.

Let $\pi\colon (X,T) \to (Y,S)$ be a factor map between TDSs. $\mathcal{U} \in \mathcal{C}_X^o$ is called *non-dense-on-fibre*, if there is $y \in Y$ such that $\pi^{-1}(y)$ is not contained in any element of $\overline{\mathcal{U}}$ which consists of the closures of elements of $\mathcal{U}$ in $X$. Clearly, if $\mathcal{U} = \{U_1,U_2\} \in \mathcal{C}_X^o$ is non-dense-on-fibre, then $\pi(U_1)\cap\pi(U_2) \neq \emptyset$.

*Definition 2.4:* Let $\pi\colon (X,T) \to (Y,S)$ be a nontrivial factor map between TDSs.

(1) Let $n \geq 2$. Say $(X,T)$ or $\pi$ has *rel.-u.p.e. of order* $n$ *(relevant to* $\pi$*)*, if any non-dense-on-fibre open cover of $X$ by $n$-sets has positive relative topological entropy. (When $n = 2$, we say simply $\pi$ or $(X,T)$ has *rel.-u.p.e.*)

(2) Say $(X,T)$ or $\pi$ has *rel.-u.p.e. of all orders (relevant to* $\pi$*)* or *relative topological* $K$ if any non-dense-on-fibre open cover of $X$ by finite sets has positive relative topological entropy, i.e. it has rel.-u.p.e. of order $n$ for any $n \geq 2$.

It is not hard to show that, for all $n \geq 2$, $\pi$ has rel.-u.p.e. of order $n$ iff $E_n(X,T|\pi) = R_\pi^{(n)} \setminus \Delta_n(X)$. Moreover, $\pi$ has relative topological $K$ iff $E_n(X,T|\pi) = R_\pi^{(n)} \setminus \Delta_n(X)$ for any $n \geq 2$.

Let $\pi_X\colon (X,T) \to (Y,S)$ and $\pi_Z\colon (Z,R) \to (Y,S)$ be two factor maps, and $\pi_1\colon X\times Z \to X$, $\pi_2\colon X\times Z \to Z$ be the projections. $J \subseteq X\times Z$ is called a *joining* of $(X,T)$ and $(Z,R)$ over $(Y,S)$ if $J$ is closed, $T\times R$-invariant with $\pi_1(J) = X$, $\pi_2(J) = Z$ and $\pi_1\times\pi_2(J) = \Delta_2(Y)$. Define

$$X\times_Y Z = \bigcup_{y\in Y} \pi_X^{-1}(y)\times\pi_Z^{-1}(y).$$

Clearly, $X \times_Y Z$ is a joining of $(X,T)$ and $(Z,R)$ over $(Y,S)$. Say $(X,T)$ and $(Z,R)$ are *disjoint* over $(Y,S)$ if $X \times_Y Z$ contains no proper sub-joining of $(X,T)$ and $(Z,R)$ over $(Y,S)$.

Let $\pi$: $(X,T) \to (Y,S)$ be a factor map. Say $\pi$ is *minimal* if $X$ is the only closed $T$-invariant subset with $\pi$-image $Y$; and $\pi$ is *open* if the $\pi$-image of any open subset is open. The proof of the following theorem is close to that of Proposition 6 in [B2].

THEOREM 2.5: *Let $\pi_X$: $(X,T) \to (Y,S)$ and $\pi_Z$: $(Z,R) \to (Y,S)$ be two factor maps with $\pi_X$ open. Suppose that $\pi_X$ has relative topological $K$, $\pi_Z$ is minimal and $h_{\text{top}}(R|\pi_Z) = 0$. Then $(X,T)$ and $(Z,R)$ are disjoint over $(Y,S)$.*

*Proof:* Let $J \subseteq X \times_Y Z$ be any given joining of $(X,T)$ and $(Z,R)$ over $(Y,S)$. Set $J(x) = \{z \in Z : (x,z) \in J\}$ for each $x \in X$. For $y \in Y$, let $J^*(y) = \bigcap_{x \in \pi_X^{-1}(y)} J(x)$. We claim

(i) $J^*(y)$ is a nonempty closed subset of $Z$ for each $y \in Y$.

(ii) $RJ^*(y) = J^*(Sy)$ for $y \in Y$.

(iii) Let $y_n \in Y$ and $z_n \in J^*(y_n)$ be such that $y_n \to y_0$ and $z_n \to z_0$ for some $y_0 \in Y$ and $z_0 \in Z$. Then $z_0 \in J^*(y_0)$.

*Proof of Claim:* (i) Fix $y \in Y$. Since $J(x)$ is compact for any $x \in X$, it remains to show $\bigcap_{i=1}^n J(x_i) \neq \emptyset$ if $x_i \in \pi_X^{-1}(y)$, $i = 1, 2, \ldots, n$. Assume the contrary, that there exists $\{x_i\}_1^n \subset \pi_X^{-1}(y)$ with $\bigcap_{i=1}^n J(x_i) = \emptyset$. Clearly, $(x_i)_1^n \notin \Delta_n(X)$. Since $\pi_X$ has relative topological $K$, it follows that $(x_i)_1^n \in E_n(X,T|\pi_X)$, and so there exists $(z_i)_1^n \in Z^{(n)}$ such that $(x_i, z_i)_1^n \in E_n(J, T \times R|\pi_X\pi_1)$ and $\pi_Z(z_1) = \cdots = \pi_Z(z_n) = y$ (by Proposition 2.3). Since $\bigcap_{i=1}^n J(x_i) = \emptyset$, we deduce that $(z_i)_1^n \notin \Delta_n(Z)$ and so $(z_i)_1^n \in E_n(Z,R|\pi_Z)$ (by Proposition 2.3 and the fact of $\pi_X\pi_1 = \pi_Z\pi_2$), a contradiction to the assumption of $h_{\text{top}}(R|\pi_Z) = 0$. This ends the proof of claim (i).

(ii) is obvious. Now we show (iii). Let $y_n \in Y$ and $z_n \in J^*(y_n)$ be such that $y_n \to y_0$ and $z_n \to z_0$. Since $\pi_X$ is an open extension, the map $y \mapsto \pi_X^{-1}(y)$ is continuous. Now we show that $z_0 \in J^*(y_0)$. For any $x_0 \in \pi_X^{-1}(y_0)$, the continuity of the map $y \mapsto \pi_X^{-1}(y)$ implies that there exists $x_n \in \pi_X^{-1}(y_n)$ with $x_n \to x_0$. Since $z_n \in J(x_n)$ and the map $x \mapsto J(x)$ is upper semi-continuous, $z_0 = \lim z_n \in J(x_0)$. Since $x_0$ is arbitrary, $z_0 \in J^*(y_0)$. **This ends the proof of the claim.**

In order to prove that $(X,T)$ and $(Z,R)$ are disjoint over $(Y,S)$, it is sufficient to prove $J = X \times_Y Z$. Put

$$J' = \bigcup_{y\in Y} \pi_X^{-1}(y) \times J^*(y) \subseteq J \subseteq X \times_Y Z.$$

By Claim (ii) and (iii), $J'$ is closed and $T \times R$-invariant. Then by claim (i), $J^*(y) \neq \emptyset$ for any $y \in Y$. Moreover $\pi_1(J') = \bigcup_{y\in Y} \pi_X^{-1}(y) = X$.

Note that $\pi_2(J')$ is a closed $R$-invariant subset of $Z$ and $\pi_Z\pi_2(J') = \pi_X\pi_1(J') = \pi_X(X) = Y$; we deduce $\pi_2(J') = Z$ from the minimality of $\pi_Z$. Since $\pi_2(J') = \bigcup_{y\in Y} J^*(y)$ and $J^*(y) \subseteq \pi_Z^{-1}(y)$, we get $J^*(y) = \pi_Z^{-1}(y)$ for all $y \in Y$. This implies

$$J' = J = \bigcup_{y\in Y} \pi_X^{-1}(y) \times \pi_Z^{-1}(y) = X \times_Y Z,$$

i.e. $(X, T)$ and $(Z, R)$ are disjoint over $(Y, S)$. ∎

*Remark 2.6:* The statement is similar to Proposition 4.2 in [GW2] but does not coincide with it. In fact, none of them is essentially contained in the other.

## 3. Relative measure-theoretical entropy tuples

In this section, the notion of relative entropy tuples in the measure-theoretical setting is introduced, basic properties are discussed and the structures of relative measure-theoretical entropy tuples are studied. Moreover, a direct proof of the lift-up property is presented.

Given a TDS $(X, T)$, we shall denote by $\mathcal{B}(X)$ the $\sigma$-algebra of Borel subsets of $X$. Let $\mathcal{M}(X)$, $\mathcal{M}(X, T)$ and $\mathcal{M}^e(X, T)$ be the collection of all probability measures on $\mathcal{B}(X)$, $T$-invariant elements of $\mathcal{M}(X)$ and ergodic elements of $\mathcal{M}(X, T)$, respectively. Then $\mathcal{M}(X)$ and $\mathcal{M}(X, T)$ are both convex, compact metric spaces when endowed with the weak$^*$-topology. Let $\mu \in \mathcal{M}(X, T)$; then $(X, \mathcal{B}(X), \mu, T)$ is a *measure-theoretical dynamical system.*

A *finite partition* of $X$ is a cover of $X$ whose elements are pairwise disjoint. Denote the set of finite partitions by $\mathcal{P}_X$. For any given $\alpha \in \mathcal{P}_X$, $\mu \in \mathcal{M}(X)$ and any sub-$\sigma$-algebra $\mathcal{C} \subseteq \mathcal{B}(X)$, set

$$H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A) \log \mu(A)$$

and

$$H_\mu(\alpha|\mathcal{C}) = \sum_{A\in\alpha} \int_X -\mathbb{E}(1_A|\mathcal{C}) \log \mathbb{E}(1_A|\mathcal{C}) d\mu,$$

where $\mathbb{E}(1_A|\mathcal{C})$ is the conditional expectation of $1_A$ with respect to $\mathcal{C}$. One standard fact states that $H_\mu(\alpha|\mathcal{C})$ increases with respect to $\alpha$ and decreases with respect to $\mathcal{C}$. Moreover, if $\mu \in \mathcal{M}(X,T)$ and $\mathcal{C}$ is $T$-invariant, then the sequence $\{H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha|\mathcal{C})\}$ is non-negative and sub-additive, and so we can define

$$h_\mu(T,\alpha|\mathcal{C}) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha|\mathcal{C}\right) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha|\mathcal{C}).$$

Now, for any given factor map, let us introduce the concept of measure-theoretical conditional entropy for all finite Borel covers (see [HYZ] for more details). Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs. For any given $\mu \in \mathcal{M}(X,T)$ and $\mathcal{U} \in \mathcal{C}_X$, define the *$\mu$-measure conditional entropy of $\mathcal{U}$ relevant to $\pi$* by

$$h_\mu(T,\mathcal{U}|\pi) = \inf_{\alpha\in\mathcal{P}_X,\ \alpha\succeq\mathcal{U}} h_\mu(T,\alpha|\pi^{-1}\mathcal{B}(Y)).$$

The *$\mu$-measure conditional entropy relevant to $\pi$* is defined by

$$h_\mu(T|\pi) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha|\pi^{-1}\mathcal{B}(Y)).$$

The relative variational principle ensures that $h_{\mathrm{top}}(T|\pi) = \sup_{\mu\in\mathcal{M}^e(X,T)} h_\mu(T|\pi)$ (see [LW], [DS] or [HYZ]).

The following results were proved in [HYZ].

THEOREM 3.1: *Let $\pi$: $X \to Y$ be a factor map between TDSs and $\mu \in \mathcal{M}(X,T)$.*

**1:** *If we set $H_\mu(\mathcal{V}|\pi) = \inf_{\alpha\in\mathcal{P}_X:\ \alpha\succeq\mathcal{V}} H_\mu(\alpha|\pi^{-1}\mathcal{B}(Y))$ for each $\mathcal{V} \in \mathcal{C}_X$, then*

$$h_\mu(T,\mathcal{U}|\pi) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}|\pi\right). \tag{3.1}$$

**2:** *Let $\mu = \int_\Omega \mu_\omega dm(\omega)$ be the ergodic decomposition of $\mu$; then*

$$h_\mu(T,\mathcal{U}|\pi) = \int_\Omega h_{\mu_\omega}(T,\mathcal{U}|\pi)dm(\omega). \tag{3.2}$$

**3:** *$h_\mu(T,\mathcal{U}|\pi) \leq h_{\mathrm{top}}(T,\mathcal{U}|\pi)$ and when $\mathcal{U} \in \mathcal{C}_X^o$*

$$\max_{\mu\in\mathcal{M}^e(X,T)} h_\mu(T,\mathcal{U}|\pi) = \max_{\mu\in\mathcal{M}(X,T)} h_\mu(T,\mathcal{U}|\pi) = h_{\mathrm{top}}(T,\mathcal{U}|\pi). \tag{3.3}$$

*Definition 3.2:* Let $\pi$: $(X,T) \to (Y,S)$ be a factor map, $\mu \in \mathcal{M}(X,T)$, $n \geq 2$ and $(x_i)_1^n \in X^{(n)} \setminus \Delta_n(X)$. $(x_i)_1^n$ is called a *relative measure-theoretical entropy n-tuple for* $\mu$ (r.m.-entropy tuple for $\mu$, for short) relevant to $\pi$, if for any admissible cover $\mathcal{U}$ with respect to $(x_i)_1^n$ we have $h_\mu(T,\mathcal{U}|\pi) > 0$.

For each $n \geq 2$, denote by $E_n^\mu(X,T|\pi)$ the set of all r.m.-entropy $n$-tuples for $\mu$ relevant to $\pi$ and write it as $E_n^\mu(X,T)$ or $E_n^\mu(X)$ when $(Y,S)$ is trivial. As in a topological setting, by similar reasoning we obtain $E_n^\mu(X,T|\pi) \subseteq R_\pi^{(n)} \setminus \Delta_n(X)$. In order to make clear the structure of $E_n^\mu(X,T|\pi)$ and the relationship between $E_n^\mu(X,T|\pi)$ and $E_n(X,T|\pi)$, we shall make some preparations. First, let us recall the concept of relative Pinsker $\sigma$-algebra which is a generalization of the classical Pinsker $\sigma$-algebra (see [Z] for more discussions).

Let $(X,T)$ be a TDS, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{A}$ a $T$-invariant sub-$\sigma$-algebra of $\mathcal{B}(X)$. The *relative Pinsker $\sigma$-algebra* $P_\mu(\mathcal{A})$ is defined as the smallest $\sigma$-algebra containing $\{\xi \in \mathcal{P}_X: h_\mu(T,\xi|\mathcal{A}) = 0\}$. Then $P_\mu(\mathcal{A})$ is a $T$-invariant $\sigma$-algebra which contains $P_\mu \cup \mathcal{A}$ where $P_\mu$ is the classical Pinsker $\sigma$-algebra of the system $(X,T)$. In fact, the classical Pinsker formula still holds for the relative Pinsker $\sigma$-algebra, and

$$P_\mu(\mathcal{A}) = \bigvee_{\xi \in \mathcal{P}_X} \bigcap_{n=0}^{+\infty} (T^{-n}\xi^- \vee \mathcal{A}). \tag{3.4}$$

Thus $P_\mu(\mathcal{A})$ is just $P_\mu$ when $\mathcal{A} = \{\emptyset, X\}$. Moreover, let $\xi \in \mathcal{P}_X$; we have

$$H_\mu(\xi|\xi^- \vee P_\mu(\mathcal{A})) = H_\mu(\xi|\xi^- \vee \mathcal{A}) = h_\mu(T,\xi|\mathcal{A}).$$

Lemma 3.3: *Let $(X,T)$ be a TDS, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{A}$ a $T$-invariant sub-$\sigma$-algebra of $\mathcal{B}(X)$. If $\xi \in \mathcal{P}_X$, then*

$$\lim_{k\to\infty} h_\mu(T^k,\xi|\mathcal{A}) = H_\mu(\xi|P_\mu(\mathcal{A})).$$

*Proof:* Obviously $\limsup_{k\to\infty} h_\mu(T^k,\xi|\mathcal{A}) \leq H_\mu(\xi|P_\mu(\mathcal{A}))$. On the other hand

$$\begin{aligned}\liminf_{k\to\infty} h_\mu(T^k,\xi|\mathcal{A}) &= \liminf_{k\to\infty} H_\mu\left(\xi|\bigvee_{n=1}^{+\infty} T^{-nk}\xi \vee \mathcal{A}\right) \\ &\geq \liminf_{n\to\infty} H_\mu(\xi|T^{-n}\xi^- \vee \mathcal{A}) \geq H_\mu\left(\xi|\bigcap_{n=0}^{+\infty}(T^{-n}\xi^- \vee \mathcal{A})\right) \\ &\geq H_\mu(\xi|P_\mu(\mathcal{A})) \quad \text{(by (3.4))}.\end{aligned}$$

That is, $\lim_{k\to\infty} h_\mu(T^k,\xi|\mathcal{A}) = H_\mu(\xi|P_\mu(\mathcal{A}))$. ∎

Let $n \in \mathbb{N}$ and $\mathcal{A}$ be a given $T$-invariant sub-$\sigma$-algebra of $\mathcal{B}(X)$; we define a new invariant measure $\lambda_n^{\mathcal{A}}(\mu)$ on the product space $X^{(n)}$ which is determined completely by

$$\lambda_n^{\mathcal{A}}(\mu)\left(\prod_{i=1}^n A_i\right) = \int_X \prod_{i=1}^n \mathbb{E}(1_{A_i}|P_\mu(\mathcal{A}))(x)d\mu(x),$$

where $A_1, \ldots, A_n \in \mathcal{B}(X)$. The following lemma is fundamental in the study of the structure of $E_n^\mu(X,T|\pi)$ which links $E_n^\mu(X,T|\pi)$ with $\lambda_n^{\mathcal{A}}(\mu)$ (letting $\mathcal{A} = \pi^{-1}\mathcal{B}(Y)$), and the proof follows the ideas of the proofs in [G1] and in Lemma 4.3 and Theorem 4.6 of [HY2].

Lemma 3.4: *Let $\pi\colon (X,T) \to (Y,S)$ be a factor map between TDSs, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{U} = \{U_1, U_2, \ldots, U_n\} \in \mathcal{C}_X$. Set $\mathcal{A} = \pi^{-1}\mathcal{B}(Y)$. Then the following are equivalent:*

**1:** $h_\mu(T,\mathcal{U}|\pi) > 0$;

**2:** *for any $\alpha \in \mathcal{P}_X$, finer than $\mathcal{U}$ as a cover, one has $h_\mu(T,\alpha|\mathcal{A}) > 0$;*

**3:** $\lambda_n^{\mathcal{A}}(\mu)(\prod_{i=1}^n U_i^c) > 0$.

*Proof:* From the definitions it is clear that $1 \Longrightarrow 2$.

Let us turn to the proof of $2 \Longrightarrow 3$. First assume that $\lambda_n^{\mathcal{A}}(\mu)(\prod_{i=1}^n U_i^c) = 0$ and $h_\mu(T,\alpha|\mathcal{A}) > 0$ for all $\alpha \in \mathcal{P}_X$, finer than $\mathcal{U}$ as a cover.

For each $1 \le i \le n$, let $C_i = \{x \in X : \mathbb{E}(1_{U_i^c}|P_\mu(\mathcal{A}))(x) > 0\} \in P_\mu(\mathcal{A})$. Since

$$\mu(U_i^c \setminus C_i) = \mu(U_i^c \cap (X \setminus C_i)) = \int_{X\setminus C_i} \mathbb{E}(1_{U_i^c}|P_\mu(\mathcal{A}))(x)d\mu(x) = 0,$$

it implies $\mu(U_i^c \setminus C_i) = 0$, and so $D_i \in P_\mu(\mathcal{A})$ and $D_i^c \subseteq U_i$; here $D_i = C_i \cup (U_i^c \setminus C_i)$. Put $D_i(0) = D_i, D_i(1) = X \setminus D_i$. Set $D_s = \bigcap_{i=1}^n D_i(s(i))$ for all $s = (s(1), s(2), \ldots, s(n)) \in \{0,1\}^n$ and $D_0^j = (\bigcap_{i=1}^n D_i) \cap (U_j \setminus \bigcup_{k=1}^{j-1} U_k)$ for $j = 1, \ldots, n$. Consider

$$\alpha = \{D_s : s \in \{0,1\}^n \quad \text{and} \quad s \neq (0,0,\ldots,0)\} \cup \{D_0^1, D_0^2, \ldots, D_0^n\} \in \mathcal{P}_X.$$

Note that as a cover, $\alpha$ is finer than $\mathcal{U}$, and so $h_\mu(T,\alpha|\mathcal{A}) > 0$.

On the other hand, by the definitions of $C_1, \ldots, C_n$, we have $\mu(\bigcap_{i=1}^n D_i) = \mu(\bigcap_{i=1}^n C_i)$ and

$$\begin{aligned} 0 = \lambda_n^{\mathcal{A}}(\mu)\left(\prod_{i=1}^n U_i^c\right) &= \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\mu(\mathcal{A}))(x)d\mu(x) \\ &= \int_{\bigcap_{i=1}^n C_i} \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\mu(\mathcal{A}))(x)d\mu(x). \end{aligned}$$

Thus, $\mu(\bigcap_{i=1}^n D_i) = \mu(\bigcap_{i=1}^n C_i) = 0$ and so $D_0^1, D_0^2, \dots, D_0^n \in P_\mu(\mathcal{A})$. Obviously, $D_s \in P_\mu(\mathcal{A})$ for each $s \in \{0,1\}^n \setminus \{(0,0,\dots,0)\}$. That is, each element of $\alpha$ belonged to $P_\mu(\mathcal{A})$, and so $h_\mu(T,\alpha|\mathcal{A}) = H_\mu(\alpha|\alpha^- \vee P_\mu(\mathcal{A})) = 0$, a contradiction.

It remains to prove $3 \Longrightarrow 1$. Since

$$\lambda_n^{\mathcal{A}}(\mu)\left(\prod_{i=1}^n U_i^c\right) = \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c}|P_\mu(\mathcal{A}))(x)d\mu(x) > 0,$$

there exists $M \in \mathbb{N}$ such that $\mu(D_M) > 0$, where

$$D_M = \{x \in X : \min_{1\le i\le n} \mathbb{E}(1_{U_i^c}|P_\mu(\mathcal{A}))(x) \ge 1/M\} \in P_\mu(\mathcal{A}).$$

For any $s = (s(1), s(2), \dots, s(n)) \in \{0,1\}^n$, set $A_s = \bigcap_{i=1}^n U_i(s(i))$ where $U_i(0) = U_i$ and $U_i(1) = X \setminus U_i$. Let $\alpha = \{A_s : s \in \{0,1\}^n\}$. Take

$$\varepsilon = \frac{\mu(D_M)}{M}\log\left(\frac{n}{n-1}\right) > 0.$$

Similar to the claim of Theorem 4.6 in [HY2], we have

Claim: *$H_\mu(\alpha|\beta \vee P_\mu(\mathcal{A})) \le H_\mu(\alpha|P_\mu(\mathcal{A})) - \varepsilon$, for all $\beta \in \mathcal{P}_X$, finer than $\mathcal{U}$ as a cover.*

Lemma 3.3 makes it possible to choose some $l \in \mathbb{N}$ with

$$h_\mu(T^l, \alpha|\mathcal{A}) > H_\mu(\alpha|P_\mu(\mathcal{A})) - \varepsilon/2. \tag{3.5}$$

Set $S = T^l$. Let $\beta \in \mathcal{P}_X$ with $\beta \succeq \mathcal{U}$. For all $k \ge 1$, one has

$$\begin{aligned} H_\mu\left(\bigvee_{i=0}^{k-1} S^{-i}\beta|\mathcal{A}\right) \ge & H_\mu\left(\bigvee_{i=0}^{k-1} S^{-i}(\beta\vee\alpha)|P_\mu(\mathcal{A})\right) \\ & - H_\mu\left(\bigvee_{i=0}^{k-1} S^{-i}\alpha|\bigvee_{i=0}^{k-1} S^{-i}\beta \vee P_\mu(\mathcal{A})\right) \\ \ge & H_\mu\left(\bigvee_{i=0}^{k-1} S^{-i}\alpha|P_\mu(\mathcal{A})\right) - \sum_{i=0}^{k-1} H_\mu(S^{-i}\alpha|S^{-i}\beta\vee P_\mu(\mathcal{A})) \\ \ge & H_\mu\left(\bigvee_{i=0}^{k-1} S^{-i}\alpha|P_\mu(\mathcal{A})\right) - k(H_\mu(\alpha|P_\mu(\mathcal{A})) - \varepsilon). \end{aligned}$$

Dividing by $k$ on both sides and letting $k \to +\infty$ we get

$$h_\mu(S,\beta|\mathcal{A}) \ge h_\mu(S,\alpha|\mathcal{A}) - H_\mu(\alpha|P_\mu(\mathcal{A})) + \varepsilon > \varepsilon/2 \quad \text{(by (3.5))}.$$

Then

$$h_\mu(T,\mathcal{U}|\pi)=\inf_{\beta\in\mathcal{P}_X,\beta\succeq\mathcal{U}} h_\mu(T,\beta|\mathcal{A})\geq \inf_{\beta\in\mathcal{P}_X,\beta\succeq\mathcal{U}}(1/\ell)h_\mu(S,\beta|\mathcal{A})\geq \varepsilon/2l>0. \qquad \blacksquare$$

*Remark 3.5:* From $1\Leftrightarrow 2$, it is easy to see that $(x_i)_1^n\in E_n^\mu(X,T|\pi)$ iff for any admissible partition $\alpha$ with respect to $(x_i)_1^n$ we have $h_\mu(T,\alpha|\pi)>0$.

For $n\geq 2$ we write $\lambda_n^\pi(\mu)=\lambda_n^{\pi^{-1}\mathcal{B}(Y)}(\mu)$. As an application of Lemma 3.4, one has

Corollary 3.6: *Let $\pi\colon (X,T)\to(Y,S)$ be a factor map between TDSs and $\mu\in\mathcal{M}(X,T)$. Then for each $n\geq 2$*

$$E_n^\mu(X,T|\pi)=\operatorname{supp}(\lambda_n^\pi(\mu))\setminus\Delta_n(X).$$

Then by using (3.2), Lemma 3.4 and Corollary 3.6, we can get the following ergodic decomposition of r.m.-entropy tuples:

Theorem 3.7: *Let $\pi\colon (X,T)\to(Y,S)$ be a factor map between TDSs and $\mu\in\mathcal{M}(X,T)$. If $\mu=\int_\Omega\mu_\omega dm(\omega)$ is the ergodic decomposition of $\mu$, then*

**1:** *$E_n^{\mu_\omega}(X,T|\pi)\subseteq E_n^\mu(X,T|\pi)$ for all $n\geq 2$ and $m$-a.e. $\omega\in\Omega$.*

**2:** *If $(x_i)_1^n\in E_n^\mu(X,T|\pi)$, then for any neighborhood $V_i$ of $x_i$ $(1\leq i\leq n)$*

$$m\Big(\Big\{\omega\in\Omega|\prod_{i=1}^n V_i\cap E_n^{\mu_\omega}(X,T|\pi)\neq\emptyset\Big\}\Big)>0.$$

*Thus for an appropriate choice of $\Omega$ we can require*

$$\overline{\bigcup\{E_n^{\mu_\omega}(X,T|\pi):\omega\in\Omega\}}\setminus\Delta_n(X)=E_n^\mu(X,T|\pi).$$

One can prove the lift-up property of r.m.-entropy tuples following the proof of Theorem 5 in [BGH]. Here we shall present a direct proof. First we need

Lemma 3.8: *Let $\pi\colon (X,T)\to(Y,S)$ be a factor map between TDSs. For each $m\in\mathcal{M}(X)$ and $\mathcal{V}\in\mathcal{C}_X$, we set $H_m(\mathcal{V})=\inf_{\alpha\in\mathcal{P}_X,\alpha\succeq\mathcal{V}}H_m(\alpha)$, $P(\mathcal{V})=\{\beta\in\mathcal{P}_X:\gamma(\mathcal{V})\succeq\beta\succeq\mathcal{V}\}$, where $\gamma(\mathcal{V})\in\mathcal{P}_X$ is the partition generated by $\mathcal{V}$. If $\mathcal{U}\in\mathcal{C}_X$, $\mu\in\mathcal{M}(X,T)$ and $\mu=\int_Y\mu_y d\nu(y)$ is the disintegration of $\mu$ over $\nu=\pi\mu\in\mathcal{M}(Y,S)$, then*

$$H_\mu(\mathcal{U}|\pi)=\int_Y H_{\mu_y}(\mathcal{U})d\nu(y)=\int_Y\min_{\alpha\in P(\mathcal{U})}H_{\mu_y}(\alpha)d\nu(y).$$

*Proof:* Let $W$ be a compact metric space and $\mathcal{W}\in\mathcal{C}_W$. Let $\gamma(\mathcal{W})$ be the partition generated by $\mathcal{W}$ and put $P(\mathcal{W})=\{\alpha\in\mathcal{P}_W:\gamma(\mathcal{W})\succeq\alpha\succeq\mathcal{W}\}$, which is a family of finite many partitions. Set $H_m(\mathcal{W}) = \inf_{\alpha\in\mathcal{P}_W:\alpha\succeq\mathcal{W}} H_m(\alpha)$ for any Borel probability measure $m$ on $W$. It is well-known that (see also the proof of Proposition 6 in [Rm])

$$H_m(\mathcal{W}) = \min_{\alpha\in P(\mathcal{W})} H_m(\alpha). \tag{3.6}$$

Assume $P(\mathcal{U}) = \{\beta_1, \beta_2, \ldots, \beta_l\}$ and put

$$A_i = \{y \in Y : H_{\mu_y}(\beta_i) = \min_{\beta\in P(\mathcal{U})} H_{\mu_y}(\beta)\}, \quad i \in \{1, 2, \ldots, l\}.$$

Let $B_1 = A_1$, $B_2 = A_2 \setminus B_1, \ldots, B_l = A_l \setminus (\bigcup_{i=1}^{l-1} B_i)$ and $B_0 = Y \setminus (\bigcup_{i=1}^{l} A_l)$.

Set $\beta^* = \{\pi^{-1}(B_0) \cap \beta_1\} \cup \{\pi^{-1}(B_i) \cap \beta_i : i = 1, \ldots, l\}$. Then $\beta^* \in \mathcal{P}_X$ and $\beta^* \succeq \mathcal{U}$. Clearly, when $i \in \{1, 2, \ldots, l\}$, for $\nu$-a.e. $y \in B_i$, $H_{\mu_y}(\beta^*) = H_{\mu_y}(\beta_i) = \min_{\beta\in P(\mathcal{U})} H_{\mu_y}(\beta) = H_{\mu_y}(\mathcal{U})$ (the latter identity follows from (3.6)). As $\nu(B_0) = 0$ (by (3.6)), we have $H_{\mu_y}(\beta^*) = H_{\mu_y}(\mathcal{U})$ for $\nu$-a.e. $y \in Y$. Moreover,

$$\begin{aligned} H_\mu(\mathcal{U}|\pi) \le H_\mu(\beta^*|\pi^{-1}\mathcal{B}(Y)) &= \int_Y H_{\mu_y}(\beta^*)d\nu(y) = \int_Y \min_{\beta\in P(\mathcal{U})} H_{\mu_y}(\beta)d\nu(y) \\ &\le \inf_{\beta\in\mathcal{P}_X:\beta\succeq\mathcal{U}} \int_Y H_{\mu_y}(\beta)d\nu(y) = \inf_{\beta\in\mathcal{P}_X:\beta\succeq\mathcal{U}} H_\mu(\beta|\pi^{-1}\mathcal{B}(Y)) = H_\mu(\mathcal{U}|\pi). \end{aligned}$$

That is, $H_\mu(\mathcal{U}|\pi) = \int_Y H_{\mu_y}(\mathcal{U})d\nu(y) = \int_Y \min_{\beta\in P(\mathcal{U})} H_{\mu_y}(\beta)d\nu(y)$. This ends the proof. ∎

Then we have

Proposition 3.9: *Let $\pi_1$: $(X, T) \to (Y, S)$ and $\pi_2$: $(Y, S) \to (Z, R)$ be two factor maps between TDSs. Suppose $n \ge 2$, $\mu \in \mathcal{M}(X, T)$ and $\nu = \pi_1\mu \in \mathcal{M}(Y, S)$. Then*

- **1:** *Suppose $(x_i)_1^n \in E_n^\mu(X, T|\pi_2\pi_1)$ and $y_i = \pi_1(x_i), i = 1, \ldots, n$. If $(y_i)_1^n \notin \Delta_n(Y)$, then $(y_i)_1^n \in E_n^\nu(Y, S|\pi_2)$.*
- **2:** *Suppose $(y_i)_1^n \in E_n^\nu(Y, S|\pi_2)$. Then there exists $(x_i)_1^n \in E_n^\mu(X, T|\pi_2\pi_1)$ such that $y_i = \pi_1(x_i), i = 1, \ldots, n$.*

*Proof:* 1 is clear by definition. Now we show 2. Assume $(y_i)_1^n \in E_n^\nu(Y, S|\pi_2)$. Let $\mathcal{V} \in \mathcal{C}_Y$ be fixed. We claim: $H_\mu(\pi_1^{-1}(\mathcal{V})|\pi_2\pi_1) = H_\nu(\mathcal{V}|\pi_2)$.

*Proof of Claim:* Set $\theta = \pi_2\nu \in \mathcal{M}(Z, R)$. Let $\mu = \int_Z \mu_z d\theta(z)$ and $\nu = \int_Z \nu_z d\theta(z)$ be the disintegration of $\mu$ and $\nu$ over $\theta$, respectively. Since $\pi_1\mu = \nu$, it is easy to see that $\pi_1\mu_z = \nu_z$ for $\theta$-a.e. $z \in Z$. Note that $P(\pi_1^{-1}\mathcal{V}) = \pi_1^{-1}P(\mathcal{V})$;

one has

$$\begin{aligned} H_\mu(\pi_1^{-1}(\mathcal{V})|\pi_2\pi_1) &= \int_Z \min_{\alpha\in P(\pi_1^{-1}\mathcal{V})} H_{\mu_z}(\alpha)d\theta(z) \quad \text{(by Lemma 3.8)} \\ &= \int_Z \min_{\beta\in P(\mathcal{V})} H_{\mu_z}(\pi_1^{-1}\beta)d\theta(z) = \int_Z \min_{\beta\in P(\mathcal{V})} H_{\pi_1\mu_z}(\beta)d\theta(z) \\ &= \int_Z \min_{\beta\in P(\mathcal{V})} H_{\nu_z}(\beta)d\theta(z) = H_\nu(\mathcal{V}|\pi_2) \quad \text{(by Lemma 3.8).} \end{aligned}$$

**This ends the proof of the claim.**

For each sufficiently large $m \in \mathbb{N}$, choose a closed neighborhood $V_i$ of $y_i$ such that the diameter of $V_i$ is at most $\frac{1}{m}$ $(i = 1, \dots, n)$ and $\bigcap_{i=1}^n V_i = \emptyset$. Set $\mathcal{V} = \{V_1^c, \dots, V_n^c\} \in \mathcal{C}_X^o$. By Claim, for $p \in \mathbb{N}$, $H_\mu(\bigvee_{i=0}^{p-1} T^{-i}\pi_1^{-1}(\mathcal{V})|\pi_2\pi_1) = H_\nu(\bigvee_{i=0}^{p-1} S^{-i}\mathcal{V}|\pi_2)$. Dividing by $p$ on both sides and letting $p \to +\infty$, one has $h_\mu(T, \pi_1^{-1}(\mathcal{V})|\pi_2\pi_1) = h_\nu(S, \mathcal{V}|\pi_2) > 0$ using (3.1) and $(y_i)_1^n \in E_n^\nu(Y, S|\pi_2)$. Then by Lemma 3.4, $\lambda_n^{\pi_2\pi_1}(\mu)(\prod_{i=1}^n \pi_1^{-1}(V_i)) > 0$. Therefore,

$$\operatorname{supp}(\lambda_n^{\pi_2\pi_1}(\mu)) \cap \prod_{i=1}^n \pi_1^{-1}(V_i) \neq \emptyset.$$

Since $\bigcap_{i=1}^n V_i = \emptyset$, one deduces

$$E_n^\mu(X, T|\pi_2\pi_1) \supseteq \operatorname{supp}(\lambda_n^{\pi_2\pi_1}(\mu)) \cap \prod_{i=1}^n \pi_1^{-1}(V_i) \neq \emptyset \quad \text{(by Corollary 3.6).}$$

Particularly, there exists $(x_i^m)_1^n \in E_n^\mu(X, T|\pi_2\pi_1)$ such that $x_i^m \in \pi_1^{-1}(V_i)$ $(i = 1, \dots, n)$. Say $(x_i)_1^n$ is a limit point of $(x_i^m)_1^n$; clearly $y_i = \pi_1(x_i)$, $i = 1, \dots, n$ and by Proposition 2.3 one has $(x_i)_1^n \in E_n^\mu(X, T|\pi_2\pi_1)$. This just completes the proof. ∎

## 4. The variational relation of relative entropy tuples and relative topological Pinsker factor

In this section, it is proved that the set of r.t.-entropy tuples is just the union of r.m.-entropy tuples over all invariant measures, and so the relative topological Pinsker factor is in fact induced by the smallest *icer* containing the set of r.t.-entropy pairs. As a by-product it turns out that an asymptotic or a distal extension does not increase topological entropy.

Theorem 4.1: *Let $\pi\colon (X, T) \to (Y, S)$ be a factor map between TDSs. Then*

**1:** *For each* $n \geq 2$ *and* $\mu \in \mathcal{M}(X,T)$

$$E_n(X,T|\pi) \supseteq E_n^{\mu}(X,T|\pi) = \operatorname{supp}(\lambda_n^{\pi}(\mu)) \setminus \triangle_n(X).$$

**2:** *There exists* $\mu \in \mathcal{M}(X,T)$ *such that for each* $n \geq 2$

$$E_n(X,T|\pi) = E_n^{\mu}(X,T|\pi)(= \operatorname{supp}(\lambda_n^{\pi}(\mu)) \setminus \triangle_n(X)).$$

*Proof:* 1. Let $(x_i)_1^n \in E_n^{\mu}(X,T|\pi) = \operatorname{supp}(\lambda_n^{\pi}(\mu)) \setminus \triangle_n(X)$. It is sufficient to show $h_{\mathrm{top}}(T,\mathcal{U}|\pi) > 0$ when $\mathcal{U} = \{U_1,\ldots,U_n\} \in \mathcal{C}_X^o$, where $U_i^c$ is a closed neighborhood of $x_i, 1 \leq i \leq n$. Since $\lambda_n^{\pi}(\mu)(\prod_{i=1}^n U_i^c) > 0$, we deduce from (3.3) and Lemma 3.4 that

$$h_{\mathrm{top}}(T,\mathcal{U}|\pi) \geq h_{\mu}(T,\mathcal{U}|\pi) > 0.$$

2. By part 1, it is enough to find some $\mu \in \mathcal{M}(X,T)$ such that $E_n(X,T|\pi) \subseteq E_n^{\mu}(X,T|\pi)$ for all $n \geq 2$. Let $n \geq 2$ be fixed. First we claim:
For any point $(x_i)_1^n \in E_n(X,T|\pi)$ with $U_i$ being the neighborhood of $x_i$, $i = 1,\ldots,n$, there exists $\nu \in \mathcal{M}(X,T)$ such that $E_n^{\nu}(X,T|\pi) \cap \prod_{i=1}^n U_i \neq \emptyset$.

*Proof of Claim:* Without loss of generality, assume that $U_i$ is a closed neighborhood of $x_i$ with $U_i \cap U_j = \emptyset$ when $x_i \neq x_j$ and $U_i = U_j$ when $x_i = x_j$, $1 \leq i < j \leq n$. Let $\mathcal{U} = \{U_1^c, U_2^c, \ldots, U_n^c\}$. Thus $h_{\mathrm{top}}(T,\mathcal{U}|\pi) > 0$. Moreover, (3.3) ensures $h_{\nu}(T,\mathcal{U}|\pi) = h_{\mathrm{top}}(T,\mathcal{U}|\pi) > 0$ for some $\nu \in \mathcal{M}(X,T)$. Then $\lambda_n^{\pi}(\nu)(\prod_{i=1}^n U_i) > 0$ (by Lemma 3.4), i.e. $\operatorname{supp}(\lambda_n^{\pi}(\nu)) \cap \prod_{i=1}^n U_i \neq \emptyset$. Since $\prod_{i=1}^n U_i \cap \Delta_n(X) = \emptyset$, from Corollary 3.6 we have $E_n^{\nu}(X,T|\pi) \cap \prod_{i=1}^n U_i \neq \emptyset$. **This ends the proof of the claim.**

Now choose a dense sequence $\{(x_i^m)_1^n : m \in \mathbb{N}\}$ in $E_n(X,T|\pi)$ with $(x_i^m)_1^n \in E_n^{\mu_n^m}(X,T|\pi)$ for some $\mu_n^m \in \mathcal{M}(X,T)$. Let $\mu = \sum_{n=2}^{+\infty} \frac{1}{2^{n-1}} (\sum_{m=1}^{+\infty} \frac{1}{2^m} \mu_n^m) \in \mathcal{M}(X,T)$. Note that if $\mu, \nu \in \mathcal{M}(X,T), a \in (0,1)$, then for any $\alpha \in \mathcal{P}_X$

$$h_{a\mu+(1-a)\nu}(T,\alpha|\pi) = a h_{\mu}(T,\alpha|\pi) + (1-a) h_{\nu}(T,\alpha|\pi). \tag{4.1}$$

Thus for $n \geq 2, m \in \mathbb{N}$, $E_n^{\mu_n^m}(X,T|\pi) \subseteq E_n^{\mu}(X,T|\pi)$ follows from (4.1). Moreover,

$$E_n(X,T|\pi) = \overline{\{(x_i^m)_1^n : m \in \mathbb{N}\}} \setminus \Delta_n(X) \subseteq \operatorname{supp}(\lambda_n^{\pi}(\mu)) \setminus \Delta_n(X) = E_n^{\mu}(X,T|\pi).$$

The conclusion is now deduced. ∎

Let $\pi\colon (X,T) \to (Y,S)$ be a factor map between TDSs. $(Z,R)$ is called *a factor of* $(X,T)$ *with respect to* $(Y,S)$ if there exist factor maps $\pi_1\colon (X,T) \to (Z,R)$

and $\pi_2$: $(Z,R) \to (Y,S)$ between TDSs such that $\pi = \pi_2 \circ \pi_1$. In this case, we say that $(Z,R)$ is *proper* if $\pi_2$ is nontrivial.

Relying on the foregoing discussions, now it is time to mention the existence of the relative topological Pinsker factor for any given factor map.

Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs. Based on the ideas of u.p.e. and c.p.e. extensions, in [GW2] Glasner and Weiss introduced the relative topological Pinsker$_1$ factor $(X_\pi,T)$ and proved that $(X_\pi,T)$ and $(Y,S)$ have the same topological entropy, where $(X_\pi,T)$ is the TDS generated by the smallest *icer* containing $E_2(X,T)\cap R_\pi$ (that is, $(X_\pi,T)$ is the "greatest" topological factor between $(X,T)$ and $(Y,S)$ whose fibers contain no entropy pairs). Later on in [LS] Lemanczyk and Siemaszko presented another approach leading to the definition of the relative topological Pinsker$_2$ factor. Let $P$ be the smallest *icer* contained in $R_\pi$ with $h_\mu(X_P,T) = h_{\pi_P(\mu)}(Y,S)$ for all $\mu \in \mathcal{M}(X_P,T)$, where $(X_P,T)$ is the system generated by $P$ and $\pi_P$: $(X_P,T) \to (Y,S)$ denotes the natural homomorphism. Moreover, it is shown in [LS] that the systems $(X_P,T)$ and $(Y,S)$ also have the same topological entropy. It turns out that $(X_\pi,T)$ is always a factor of $(X_P,T)$. Just after [LS], in [PS] Park and Siemaszko interpreted the relative topological Pinsker$_2$ factor using relative measure-theoretical entropy and proved that $P$ was just generated by $\bigcup_{\mu\in\mathcal{M}(X,T)}\mathrm{supp}(\lambda_2^\pi(\mu))$. Thus by Corollary 3.6 and Theorem 4.1, the relative topological Pinsker$_2$ factor is just the factor induced by the smallest *icer* containing $E_2(X,T|\pi)$, the set of r.t.-entropy pairs. That is, $(X_P,T)$ is the "greatest" topological factor between $X$ and $Y$ whose fibers contain no r.t.-entropy pairs. It implies that $h_{\mathrm{top}}(X_P|\pi_P) = 0$, and consequently $(X_P,T)$ and $(Y,S)$ have the same topological entropy.

By using some known theorems we can show (see, for example, [HY2], or Lemma 3.1 in [DYZ])

PROPOSITION 4.2: *Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs and $\mu \in \mathcal{M}^e(X,T)$ with $h_\mu(T|\pi) > 0$. Then $\lambda_n^\pi(\mu)$ is ergodic and $\lambda_n^\pi(\mu)(\Delta_n(X)) = 0$, $\forall n \geq 2$.*

*Remark 4.3:* For each $n \geq 2$, $(x_i)_1^n \in E_n(X,T|\pi)$ is called *intrinsic* if $x_i \neq x_j$ when $i \neq j$. Denote by $E_n^e(X,T|\pi)$ the set of all intrinsic r.t.-entropy $n$-tuples. Then following the proof of Theorem 6.5 in [HY2] and using the corresponding results obtained in this paper (including Proposition 4.2) one can show that $E_n(X,T|\pi)=\overline{E_n^e(X,T|\pi)} \setminus \Delta_n(X)$ for all $n \geq 2$. In particular, if $h_{\mathrm{top}}(T|\pi) > 0$, then $E_n^e(X,T|\pi) \neq \emptyset$ for each $n \geq 2$. Moreover, following the proof of Lemma 4.1 in [DYZ], one can prove that there is an uncountable subset $K$ of $X$ such that each tuple from $K$ is a r.t.-entropy tuple.

We now apply Proposition 4.2 to show that an asymptotic or a distal extension does not increase entropy. First we need some notions. Let $(X,T)$ be a TDS and $d$ be the metric on $X$. A pair $(x,y) \in X^{(2)}$ is called *asymptotic* (resp. *distal*) if $\lim_{n\to+\infty} d(T^n x, T^n y) = 0$ (resp. $\liminf_{n\to+\infty} d(T^n x, T^n y) > 0$) and the pair $(x,y)$ is *proper* if $x \neq y$. Denote by $AP(X,T)$ and $D(X,T)$ the set of all asymptotic pairs and distal pairs of $(X,T)$, respectively; then $AP(X,T)$ is a $T\times T$-invariant Borel subset of $X^{(2)}$. A factor map $\pi$: $X \to Y$ is *asymptotic* (resp. *distal*) if $R_\pi \subset AP(X,T)$ (resp. $R_\pi \setminus \Delta_2(X) \subset D(X,T)$). We have

THEOREM 4.4: *Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs. If $\pi$ is asymptotic or distal, then $h_{\rm top}(T|\pi) = 0$ and thus $h_{\rm top}(T) = h_{\rm top}(S)$.*

*Proof:* It follows from Theorem 4.2 in [Z]. But here we present a direct proof of it using Proposition 4.2.

Assume first that $\pi$ is asymptotic. Then $R_\pi \subset AP(X,T)$, and so $E_2(X,T|\pi) \subset R_\pi \subset AP(X,T)$. We claim that $E_2(X,T|\pi) = \emptyset$. Assume the contrary that $E_2(X,T|\pi) \neq \emptyset$. Then the relative variational principle ensures that there is $\mu \in \mathcal{M}^e(X,T)$ with $h_\mu(T|\pi) > 0$. It follows by Proposition 4.2 that $\lambda_2^\pi(\mu)$ is ergodic and $\lambda_2^\pi(\mu)(\Delta_2(X)) = 0$. Thus, $\mathrm{supp}(\lambda_2^\pi(\mu))$ is a transitive system, properly contains $\Delta_2(X)$. This implies that there is a pair in $\mathrm{supp}(\lambda_2^\pi(\mu)) \setminus \Delta_2(X) \subset E_2(X,T|\pi)$ which is not asymptotic, a contradiction. So $E_2(X,T|\pi) = \emptyset$. It follows that $h_{\rm top}(T|\pi) = 0$ and $h_{\rm top}(T) = h_{\rm top}(S)$.

When $\pi$ is distal, the same proof works. ∎

## 5. Relative c.p.e. extensions

In this section, we shall define rel.-c.p.e. extension based on the notion of r.t.-entropy pairs. The main result of this section is Theorem 5.4, which says roughly that for a rel.-c.p.e. extension, each fibre either is a singleton or is fully supported.

*Definition 5.1:* Let $\pi$: $(X,T) \to (Y,S)$ be a nontrivial factor map between TDSs. We say $\pi$ has *relative completely positive entropy* (for short *rel.-c.p.e.*) if $(Z,R)$ has positive relative topological entropy with respect to $(Y,S)$ for any proper factor $(Z,R)$ of $(X,T)$ with respect to $(Y,S)$. In this case, we also say that $(X,T)$ has *rel.-c.p.e.* with respect to $(Y,S)$.

*Remark 5.2:* It is clear that rel.-u.p.e. implies rel.-c.p.e. Properties of rel. -u.p.e. and rel.-c.p.e. are both stable under factor maps. Precisely, let

$\pi$: $(X,T) \to (Y,S)$ be a given factor map between TDSs, if $\pi$ has property P (P denotes one of the properties: rel.-u.p.e. and rel.-c.p.e.) and $(Z,R)$ is a proper factor of $(X,T)$ with respect to $(Y,S)$; then $(Z,R)$ also has property P with respect to $(Y,S)$. Note that rel.-u.p.e. (resp. rel.-c.p.e.) recovers u.p.e. (resp. c.p.e.) introduced in [B1], when $(Y,S)$ is trivial.

A rel.-c.p.e. extension can be characterized as follows.

PROPOSITION 5.3: *Let $\pi$: $(X,T) \to (Y,S)$ be a nontrivial factor map between TDSs. Then $(X,T)$ has rel.-c.p.e. with respect to $(Y,S)$ iff $R_\pi$ coincides with the icer generated by $E_2(X,T|\pi) \cup \Delta_2(X)$.*

*Proof:* Let $R$ be the *icer* generated by $E_2(X,T|\pi) \cup \Delta_2(X)$. Clearly, $R \subseteq R_\pi$. Let $\pi_1$: $(X,T) \to (Z,\Theta)$ be the factor map determined by $R$. Since $R \subseteq R_\pi$, there exists an extension $\pi_2$: $(Z,\Theta) \to (Y,S)$ such that $\pi_2\pi_1 = \pi$. By Proposition 2.3, $h_{\mathrm{top}}(\theta|\pi_2) = 0$. Hence if $(X,T)$ has rel.-c.p.e. with respect to $(Y,S)$, then $\pi_2$ is trivial and so $R = R_\pi$.

Conversely, assume $R = R_\pi$. Then $\pi_1 = \pi$. For any proper factor $(Z,R)$ of $(X,T)$ with respect to $(Y,S)$, let $\theta_1$: $(X,T) \to (Z,R)$ and $\theta_2$: $(Z,R) \to (Y,S)$ with $\theta_2\theta_1 = \pi$. Now we show that $h_{\mathrm{top}}(R|\theta_2) > 0$, which implies $\pi$ has rel.-c.p.e. If not, then we have $h_{\mathrm{top}}(R|\theta_2) = 0$. Hence $E_2(Z,R|\theta_2) = \emptyset$. By Proposition 2.3, $\theta_1 \times \theta_1(E_2(X,T|\pi)) \subseteq \Delta_2(Z)$. Hence $E_2(X,T|\pi) \cup \Delta_2(X) \subseteq R_{\theta_1}$. Since $R_{\theta_1}$ is an *icer* on $X$, so $R_{\theta_1} \supseteq R$. This implies $R_{\theta_1} = R_\pi$, i.e. $\theta_1 = \pi$. Moreover, $\theta_2$ is one-to-one, a contradiction. ∎

Let $(X,T)$ be a TDS and $\mu \in \mathcal{M}(X,T)$. Recall that $\mathrm{supp}(\mu)$ stands for the support of $\mu$, and when $\mathrm{supp}(\mu) = X$, we say $\mu$ *has full support.* Denote $\bigcup_{\mu\in\mathcal{M}(X,T)} \mathrm{supp}(\mu)$ by $\mathrm{supp}(X,T)$, and note that there is $\mu \in \mathcal{M}(X,T)$ with $\mathrm{supp}(X,T) = \mathrm{supp}(\mu)$. It is known that if $(X,T)$ has c.p.e., then for any nonempty open subset $U \subseteq X$ there exists $\mu \in \mathcal{M}(X,T)$ such that $\mu(U) > 0$, and so $\mathrm{supp}(X,T) = X$ (see Corollary 7 of [B1]). In the following, we shall generalize the result to the relative case.

Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs. Then $\pi$ induces $\pi^*$: $\mathcal{M}(X,T) \to \mathcal{M}(Y,S)$ where $\pi^*(\mu) = \pi\mu$. Moreover, $\pi^*$ is surjective and for any $\mu \in \mathcal{M}(X,T)$, $\mathrm{supp}(\pi\mu) = \pi(\mathrm{supp}(\mu))$ and $\pi^{-1}(\mathrm{supp}(\pi\mu)) \supseteq \mathrm{supp}(\mu)$. Similarly, $\mathrm{supp}(Y,S) = \pi(\mathrm{supp}(X,T))$ and $\pi^{-1}(\mathrm{supp}(Y,S)) \supseteq \mathrm{supp}(X,T)$. In general, the identity doesn't hold. But if the factor map $\pi$ has some good properties (for example rel.-c.p.e.), then the identity holds. Before stating Theorem 5.4, we recall some notations.

Let $(X,T)$ be a TDS. $(X,T)$ is called *transitive* if for any given pair $(U,V)$ of nonempty open subsets of $X$, $N(U,V)=\{n\in\mathbb{Z}_+ : U\cap T^{-n}V\neq\emptyset\}$ is nonempty. $x\in X$ is called a *transitive point* of $(X,T)$ if the *orbit* of $x$, $\{x,Tx,T^2x,\dots\}$, is dense in $X$. Denote by $Tran(X,T)$ the set of all transitive points. It is well-known that the system $(X,T)$ is transitive iff $Tran(X,T)$ forms a dense $G_\delta$ subset of $X$. $(X,T)$ is called *minimal* if $Tran(X,T)=X$, that is, each orbit is dense in the space. $x\in X$ is called a *minimal point* of $(X,T)$ if the sub-system $\overline{\{x,Tx,T^2x,\dots\}}$ is minimal. $(X,T)$ is called *weakly mixing* if $(X\times X, T\times T)$ is transitive. Let $\pi$: $(X,T)\to(Y,S)$ be a factor map between TDSs. We say $\pi$ is *weakly mixing* (or call $(X,T)$ a *weakly mixing extension* of $(Y,S)$) if $(R_\pi, T\times T)$ is transitive, and call $\pi$ *weakly mixing of all orders* if $(R_\pi^{(n)}, T^{(n)})$ is transitive for each $n\geq 2$.

THEOREM 5.4: *Let $\pi$: $(X,T)\to(Y,S)$ be a factor map between TDSs. Suppose that $\pi$ has rel.-c.p.e. Then*

**1:** $\pi^{-1}(\mathrm{supp}(Y,S))=\mathrm{supp}(X,T)$, *i.e.*

$$\forall y\in \mathrm{supp}(Y,S), \pi^{-1}(y)\subset \mathrm{supp}(X,T).$$

**2:** *For each $y\in Y\setminus\mathrm{supp}(Y,S)$, $\pi^{-1}(y)$ is a singleton.*

**3:** *Consequently, $X$ is fully supported iff so is $Y$, and $X$ is fully supported when $\pi$ is weakly mixing.*

We remark that the second situation can occur; see, for example, Remark 6.3 (assuming that $(Y,S)$ is supported on the fixed point $p$). The proof of Theorem 5.4 is completed by the following two lemmas.

LEMMA 5.5: *Let $\pi$: $(X,T)\to(Y,S)$ be a factor map between TDSs and $\pi$ has rel.-c.p.e. Suppose that $\mu\in\mathcal{M}(X,T)$ satisfies $E_2^\mu(X,T|\pi)=E_2(X,T|\pi)$. Then $\pi^{-1}(y)$ is a singleton for all $y\in Y\setminus\mathrm{supp}(\pi\mu)$.*

*Proof:* Let $(x_1,x_2)\in E_2(X,T|\pi)$ and $y=\pi(x_1)=\pi(x_2)$. We claim that $y\in\mathrm{supp}(\pi\mu)$. Assume the contrary that $y\in Y\setminus\mathrm{supp}(\pi\mu)$, i.e. $x_1,x_2\notin\mathrm{supp}(\mu)$. Then we can find closed neighborhoods $U_i$ of $x_i$ such that $U_1\cap U_2=\emptyset$ and $\mu(U_i)=0,\ i=1,2$. Let $\alpha=\{U_1,U_1^c\}$. It is clear that $\alpha$ is an admissible partition with respect to $(x_1,x_2)$ and $h_\mu(T,\alpha|\pi)=0$. Hence $(x_1,x_2)\notin E_2^\mu(X,T|\pi)=E_2(X,T|\pi)$, a contradiction.

Let $R=\bigcup_{y\in\mathrm{supp}(\pi\mu)}(\pi^{-1}(y)\times\pi^{-1}(y))\cup\Delta_2(X)$. By what we just proved it follows that $E_2(X,T|\pi)\cup\Delta_2(X)\subseteq R$. Note that $R\subseteq R_\pi$ is an *icer* on $X$. So $R$ contains the *icer* generated by $E_2(X,T|\pi)\cup\Delta_2(X)$, and so $R=R_\pi$ by

Proposition 5.3, as $\pi$ has rel.-c.p.e. This implies that

$$\bigcup_{y\in Y\setminus \mathrm{supp}(\pi\mu)} (\pi^{-1}(y)\times\pi^{-1}(y)) \subseteq R\setminus \bigcup_{y\in \mathrm{supp}(\pi\mu)} (\pi^{-1}(y)\times\pi^{-1}(y)) \subseteq \Delta_2(X),$$

i.e. $\pi^{-1}(y)$ contains only one point for all $y\in Y\setminus \mathrm{supp}(\pi\mu)$. ∎

Lemma 5.6: *Let* $\pi\colon (X,T)\to(Y,S)$ *be a factor map between TDSs and* $\pi$ *has rel.-c.p.e. Suppose that* $\mu\in\mathcal{M}(X,T)$ *satisfies* $E_2^{\mu}(X,T|\pi)=E_2(X,T|\pi)$. *Then* $\mathrm{supp}(\mu)=\pi^{-1}(\mathrm{supp}(\pi\mu))$. *In particular,* $\mu$ *has full support iff* $\pi\mu$ *has full support.*

*Proof:* The idea of the proof is inspired by that of Proposition 6 in [B1].

Suppose the contrary that there exists $x_0\in\pi^{-1}(\mathrm{supp}(\pi\mu))\setminus\mathrm{supp}(\mu)$. Take an open neighborhood $V$ of $x_0$ with $\mu(V)=0$ and put $U=\bigcup_{n\in\mathbb{Z}}T^n(V)$. Then $U$ is open with $\mu(U)=0$. Clearly, $U\cap\mathrm{supp}(\mu)=\emptyset$, i.e. $\mathrm{supp}(\mu)\subseteq U^c$. Since $\pi(x_0)\in\mathrm{supp}(\pi\mu)=\pi(\mathrm{supp}(\mu))\subseteq\pi(U^c)$, there is $x_1\in U^c$ with $\pi(x_1)=\pi(x_0)$.

Using Urysohn's Lemma, we construct a continuous map $f\colon X\to[0,1]$ such that $f(x_0)=0$ and $f(x)=1$ on $U^c$. Define $F\colon X\to[0,1]^{\mathbb{Z}}$ by

$$(F(x))_n=f(T^nx)\quad\text{for all } x\in X \text{ and } n\in\mathbb{Z}.$$

Define $\pi_1\colon X\to[0,1]^{\mathbb{Z}}\times Y$ by letting $\pi_1(x)=(F(x),\pi(x))$ for all $x\in X$. Now endow $W=\pi_1(X)$ with the transformation $\sigma\times S$, where $\sigma$ is the shift map on $[0,1]^{\mathbb{Z}}$, i.e. $\sigma((z_n)_{n\in\mathbb{Z}})=(z_{n+1})_{n\in\mathbb{Z}}$ for $(z_n)_{n\in\mathbb{Z}}\in[0,1]^{\mathbb{Z}}$. It is easy to see that $(W,\sigma\times S)$ is a TDS and $\pi_1\colon(X,T)\to(W,\sigma\times S)$ is a factor map between TDSs.

Let $\pi_2\colon W\to Y$ be the projection to the second coordinate. Then

$$\pi_2\colon (W,\sigma\times S)\to(Y,S)$$

is a factor map and $\pi=\pi_2\pi_1$. Since $U^c$ is $T$-invariant, $F(U^c)=\{1^{\mathbb{Z}}\}$. Particularly, $F(x_1)=1^{\mathbb{Z}}\neq F(x_0)$. Moreover, $\pi_1(x_0)\neq\pi_1(x_1)$. Combining this fact and the equality $\pi_2(\pi_1(x_0))=\pi(x_0)=\pi(x_1)=\pi_2(\pi_1(x_1))$, one knows that $\pi_2$ is a nontrivial extension.

Since $\pi$ has rel.-c.p.e., $\pi_2$ is a nontrivial rel.-c.p.e. extension. By Propostion 5.3, $R_{\pi_2}$ is the smallest *icer* on $W$ generated by $E_2(W,\sigma\times S|\pi_2)\cup\Delta_2(W)$. Since $R_{\pi_2}\neq\Delta_2(W)$ and $\Delta_2(W)$ is an *icer* on $W$, $E_2(W,\sigma\times S|\pi_2)\neq\emptyset$. Note that

$$\begin{aligned}E_2^{\pi_1\mu}(W,\sigma\times S|\pi_2)&=\pi_1(E_2^{\mu}(X,T|\pi))\setminus\Delta_2(X)\quad\text{(by Proposition 3.9)}\\&=\pi_1(E_2(X,T|\pi))\setminus\Delta_2(X)\\&=E_2(W,\sigma\times S|\pi_2)\quad\text{(by Proposition 2.3)}\\&\neq\emptyset,\end{aligned}$$

there exist $y \in Y$ and $w_1 \neq w_2 \in \pi_2^{-1}(y)$ such that $(w_1, w_2) \in E_2^{\pi_1\mu}(W, \sigma\times S|\pi_2)$. Without loss of generality, we assume that $w_1 \neq (1^{\mathbb{Z}}, y)$, i.e. $w_1 \notin \{1^{\mathbb{Z}}\} \times Y$. Since $F(U^c) = \{1^{\mathbb{Z}}\}$, $\pi_1(U^c) \subseteq \{1^{\mathbb{Z}}\} \times Y$. Moreover,

$$\mathrm{supp}(\pi_1\mu) = \pi_1(\mathrm{supp}(\mu)) \subseteq \pi_1(U^c) \subseteq \{1^{\mathbb{Z}}\} \times Y;$$

this implies that $w_1 \notin \mathrm{supp}(\pi_1\mu)$. Thus we can find closed neighborhoods $U_i$ of $w_i$ such that $U_1 \cap U_2 = \emptyset$ and $\pi_1\mu(U_1) = 0$. Let $\alpha = \{U_1, U_1^c\}$. Then $\alpha$ is an admissible partition with respect to $(w_1, w_2)$. Since $\pi_1\mu(U_1) = 0$, $h_{\pi_1\mu}(T, \alpha|\pi_2) = 0$. Hence $(w_1, w_2) \notin E_2^{\pi_1\mu}(W, \sigma \times S|\pi_2)$, a contradiction. ∎

Now we are ready to prove Theorem 5.4.

*Proof of Theorem 5.4:* By Theorem 4.1, there exists $\mu_1 \in \mathcal{M}(X,T)$ with $E_2^{\mu_1}(X, T|\pi) = E_2(X, T|\pi)$. Clearly, there exists $\mu_2 \in \mathcal{M}(X,T)$ such that $\mathrm{supp}(\pi\mu_2) = \mathrm{supp}(Y, S)$. Let $\mu = \frac{1}{2}(\mu_1 + \mu_2)$. Then $\mu \in \mathcal{M}(X,T)$, $\mathrm{supp}(\pi\mu) \supseteq \mathrm{supp}(\pi\mu_2)$ and $E_2^{\mu}(X, T|\pi) \supseteq E_2^{\mu_1}(X, T|\pi)$; the last inequality comes from the fact that for any $\alpha \in \mathcal{P}_X$, the relative entropy map $\nu \in \mathcal{M}(X,T) \mapsto h_\nu(T, \alpha|\pi)$ is an affine map. So $\mathrm{supp}(\pi\mu) = \mathrm{supp}(Y, S)$ and $E_2^{\mu}(X, T|\pi) = E_2(X, T|\pi)$. Therefore by Lemma 5.5 and Lemma 5.6, 1 and 2 are proved. In particular, $X$ is fully supported iff so is $Y$.

To end the proof assume that $\pi$ is weakly mixing. By 1, it suffices to show $\mathrm{supp}(Y, S) = Y$. Since $\pi$ has rel.-c.p.e., it is nontrivial. If $\mathrm{supp}(Y, S) \subsetneq Y$, using 1 and 2 we have

$$\{(x, x) : x \in X \setminus \mathrm{supp}(X, T)\} = (\pi \times \pi)^{-1}(\{(y, y) : y \in Y \setminus \mathrm{supp}(Y, S)\})$$

is a nonempty open subset of $R_\pi$, and so $R_\pi = \Delta_2(X)$ (as $\{(x, x) : x \in X \setminus \mathrm{supp}(X, T)\} \subseteq \Delta_2(X)$ and $\pi$ is weakly mixing), i.e. $\pi$ is a trivial extension, a contradiction. ∎

Let $\pi$: $(X, T) \to (Y, S)$ be a nontrivial factor map between TDSs. Recall that in [GW2] $\pi$ is called an *entropy-generated extension* if $R_\pi$ coincides with the *icer* generated by $((E_2(X, T) \cup \Delta_2(X)) \cap R_\pi$. As $E_2(X, T|\pi) \subseteq E_2(X, T)$, by Proposition 5.3 if $\pi$ has rel.-c.p.e. then $\pi$ is an entropy-generated extension.

Let $(Z, R)$ be a proper factor of $(X, T)$ with respect to $(Y, S)$ with $\pi = \pi_2 \circ \pi_1$, where $\pi_1$: $(X, T) \to (Z, R)$ and $\pi_2$: $(Z, R) \to (Y, S)$. Denote by $I_{\pi_2}$ the *icer* generated by $((E_2(Z, R) \cup \Delta_2(Z)) \cap R_{\pi_2}$. It is clear that $(\pi_1 \times \pi_1)^{-1}(I_{\pi_2})$ forms an *icer* containing $((E_2(X, T) \cup \Delta_2(X)) \cap R_\pi$. Then $\pi_2$ is also an entropy-generated extension if $\pi$ is an entropy-generated extension. Thus following the same idea as in Theorem 5.4, Lemma 5.5 and Lemma 5.6 we have the following result, which is stronger than Theorem 5.4.

THEOREM 5.7: *Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs. Suppose that $\pi$ is an entropy-generated extension. Then*

**1:** *Suppose that $\mu \in \mathcal{M}(X,T)$ satisfies $E_2^{\mu}(X,T) \cap R_\pi = E_2(X,T) \cap R_\pi$. Then* $\mathrm{supp}(\mu) = \pi^{-1}(\mathrm{supp}(\pi\mu))$ *and $\pi^{-1}(y)$ is a singleton ($\forall y \notin \mathrm{supp}(\pi\mu)$).*

**2:** $\mathrm{supp}(X,T) = \pi^{-1}(\mathrm{supp}(Y,S))$ *and $\pi^{-1}(y)$ is a singleton* ($\forall y \notin \mathrm{supp}(Y,S)$).

**3:** *If $\pi$ is weakly mixing, then* $\mathrm{supp}(X,T) = X$.

A TDS $(X,T)$ is called an *$E$-system* if it is transitive and $\mathrm{supp}(X,T) = X$. It is well-known that a minimal system is an $E$-system and the product of an $E$-system with a weakly mixing system is transitive (see [GW1, GW3]). As a direct corollary of Theorem 5.7, we have

COROLLARY 5.8: *Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between transitive TDSs and $\pi$ an entropy-generated extension. Then $(X,T)$ is an $E$-system iff so is $(Y,S)$.*

## 6. Relative u.p.e. extensions

An interesting result obtained in [B1] is that u.p.e. implies weak mixing, which makes clearer how topological entropy is woven into the general pattern of topological dynamics. Does the corresponding result hold in some general setting, which is a natural question that arises immediately in one's mind when thinking about the relative case? In this section, we will give a partial answer to the question using the ideas and results obtained in [G1] and [PS].

First, let us recall a definition which appeared first in [GW2] as a generalization of u.p.e. which is different from ours in the paper. Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs. $\pi$ is called a *u.p.e. extension* if $R_\pi \setminus \Delta_2(X) \subseteq E_2(X,T)$. Note that $E_2(X,T|\pi) \subseteq E_2(X,T)$, and hence $\pi$ is a u.p.e. extension when $\pi$ has rel.-u.p.e. The following lemma comes from Remark of Lemma 4.8 and Corollary 5.7 in [HSY].

LEMMA 6.1: *Let $(X,T)$ be a TDS. If $(x_1,x_2) \in E_2(X,T)$, then for any infinite sequence $F$ of $\mathbb{Z}_+$ and any open neighborhood $U_i$ of $x_i$, $i=1,2$, one has $N(U_1,U_2) \cap (F-F) \neq \emptyset$, where $F-F = \{a-b \geq 0 : a,b \in F\}$.*

Let $\pi$: $(X,T) \to (Y,S)$ be a factor map between TDSs. We say $\pi$ is *semi-open*, if for any nonempty open subset $U$, $\mathrm{int}(\pi(U)) \neq \emptyset$. Any factor map

$\pi$: $(X, T) \to (Y, S)$, where $X$ is minimal, is semi-open. The following lemma may be viewed as the motivation of this section.

LEMMA 6.2: *Let* $\pi$: $(X, T) \to (Y, S)$ *be a semi-open factor map between TDSs and* $\pi$ *be a u.p.e. extension. Then* $(Z \times X, R \times T)$ *is transitive iff* $(Z \times Y, R \times S)$ *is transitive, where* $(Z, R)$ *is a TDS.*

*Proof:* It remains to show the transitivity of $R \times S$ implies that of $R \times T$. To do this, it suffices to prove that for any nonempty open sets $U_1, U_2$ of $Z$ and $V_1, V_2$ of $X$, $N(U_1 \times V_1, U_2 \times V_2) \neq \emptyset$.

Let $U_1, U_2$ and $V_1, V_2$ be the subsets mentioned above. Since $\pi$ is semi-open, $W_i := \text{int}(\pi(V_i)) \neq \emptyset, i = 1, 2$. Moreover, there exists $n \in \mathbb{N}$ such that $U_1 \times W_1 \cap R^{-n}U_2 \times S^{-n}W_2 \neq \emptyset$, as $R \times S$ is transitive. Put $U = U_1 \cap R^{-n}U_2$. Then $U$ is a nonempty open subset of $Z$. Take $z \in U \cap Tran(Z, R)$ ($R \times S$ is transitive). Then $F := N(z, U) = \{i \in \mathbb{Z}_+ : R^i(z) \in U\}$ is infinite. It is clear that $N(U_1, R^{-n}U_2) \supseteq N(U, U) = F - F$.

Since $W_1 \cap S^{-n}W_2 \neq \emptyset$ and $S^{-n}\pi(V_2) = \pi(T^{-n}V_2)$, we have

$$\pi(V_1) \cap \pi(T^{-n}V_2) \neq \emptyset.$$

There are two cases:

CASE 1: $V_1 \cap T^{-n}V_2 \neq \emptyset$. In this case $n \in N(U_1 \times V_1, U_2 \times V_2) \neq \emptyset$.

CASE 2: $V_1 \cap T^{-n}(V_2) = \emptyset$. Since $\pi(V_1) \cap \pi(T^{-n}V_2) \neq \emptyset$, there exist $x_1 \in V_1$ and $x_2 \in T^{-n}V_2$ with $\pi(x_1) = \pi(x_2)$. Since $\pi$ is a u.p.e. extension, $(x_1, x_2) \in E_2(X, T)$. By Lemma 6.1, $N(V_1, T^{-n}V_2) \cap (F - F) \neq \emptyset$. Hence $N(U_1 \times V_1, R^{-n}U_2 \times T^{-n}V_2) = N(V_1, T^{-n}V_2) \cap N(U_1, R^{-n}U_2) \neq \emptyset$. Note that $N(U_1 \times V_1, U_2 \times V_2) \supseteq N(U_1 \times V_1, R^{-n}U_2 \times T^{-n}V_2) + n$. Thus, $N(U_1 \times V_1, U_2 \times V_2) \neq \emptyset$. This finishes the proof. ∎

*Remark 6.3:* The assumption of semi-openness is necessary in the above lemma. For example, let $(Y, S)$ be a nontrivial transitive system having a fixed point $p$ and $(Z, R) = (\{0, 1\}^{\mathbb{Z}}, \sigma)$ be the full shift on two symbols. By identifying the point $0^{\mathbb{Z}}$ in $Z$ with the point $p$ in $Y$ we get a new compact metric space $X$. There exists a natural transformation $T$ on $X$ such that $T|_Y = S$ and $T|_{\{0,1\}^{\mathbb{Z}}} = \sigma$. Clearly, $(X, T)$ is not transitive. Now let $\pi$: $X \to Y$ with $\pi(x) = p$ if $x \in \{0, 1\}^{\mathbb{Z}}$ and $\pi(x) = x$ if $x \in Y$. Then $\pi$: $(X, T) \to (Y, S)$ is a factor map and it is easy to see that $\pi$ has rel.-u.p.e., as $(\{0, 1\}^{\mathbb{Z}}, \sigma)$ has u.p.e.

Let $(X,T)$ be a TDS. We say $(X,T)$ is *mildly mixing* if its product with any transitive system is transitive (see, for example, [GW4], [HY1]). As an application of Lemma 6.2, it is not hard to obtain

Corollary 6.4: *The following two statements hold.*

**1:** *Let* $\pi: (X,T) \to (Y,S)$ *be a semi-open factor map between TDSs. If* $\pi$ *is a u.p.e. extension, then* $(X,T)$ *is transitive (resp. weakly mixing, mildly mixing) iff* $(Y,S)$ *is transitive (resp. weakly mixing, mildly mixing).*

**2:** *In general, let* $\pi_i: (X_i,T_i) \to (Y_i,S_i)$ $(1 \le i \le n)$ *be* $n$ *given u.p.e. extensions between TDSs. If all* $\pi_i$ $(1 \le i \le n)$ *are semi-open, then* $(X_1 \times \cdots \times X_n, T_1 \times \cdots \times T_n)$ *is transitive (resp. weakly mixing, mildly mixing) iff* $(Y_1 \times \cdots \times Y_n, S_1 \times \cdots \times S_n)$ *is transitive (resp. weakly mixing, mildly mixing).*

*Proof:* We only prove 1 in the situation when $(Y,S)$ is weakly mixing, and the other cases can be proved in a similar way. In the case $S \times S$ is transitive, and hence by Lemma 6.2, we know that $S \times T$ and consequently $T \times S$ is transitive. Using Lemma 6.2 again we get that $T \times T$ is transitive, i.e. $T$ is weakly mixing. ∎

By the definitions, it is clear that if $\pi$ is a u.p.e. extension then $\pi$ is an entropy-generated extension. Then with the help of Theorem 5.3, using Lemma 6.2 we are able to give a similar result as Corollary 5.8 directly.

Corollary 6.5: *Let* $\pi: (X,T) \to (Y,S)$ *be a semi-open factor map between TDSs and* $\pi$ *be a u.p.e. extension. Then* $(X,T)$ *is an E-system iff so is* $(Y,S)$.

In the remaining part of the section we focus on the question when a rel.-u.p.e. extension is a weakly mixing one. Let $(X,T)$ be a TDS and $\mu \in \mathcal{M}(X,T)$. Denote

$$s_\mu^2 = (\mathrm{supp}(\mu) \times \mathrm{supp}(\mu)) \cap \Delta_2(X), s_X^2 = (\mathrm{supp}(X,T) \times \mathrm{supp}(X,T)) \cap \Delta_2(X).$$

Let $\pi: (X,T) \to (Y,S)$ be a factor map between TDSs; we shall write $E_2^\mu(\pi) = E_2^\mu(X,T|\pi)$ for short if there is no ambiguity.

Let $\pi_i: (X_i,T_i) \to (Y,S)$ $(i=1,2)$ be two factor maps between TDSs. Recall that

$$X_1 \times_Y X_2 = \{(x_1,x_2) \in X_1 \times X_2 : \pi_1(x_1) = \pi_2(x_2)\} \quad \text{and}$$
$$T_1 \times_Y T_2: X_1 \times_Y X_2 \to X_1 \times_Y X_2 \quad \text{with } (x_1,x_2) \mapsto (T_1x_1, T_2x_2).$$

Then $(X_1 \times_Y X_2, T_1 \times_Y T_2)$ forms a TDS. Denote by $\pi_1 \times_Y \pi_2$ the factor map induced naturally by $\pi_1$ and $\pi_2$, i.e.

$$\pi_1 \times_Y \pi_2\colon (X_1 \times_Y X_2, T_1 \times_Y T_2) \to (Y, S) \quad \text{with } (x_1, x_2) \mapsto \pi_1(x_1).$$

Assume $\mu_i \in \mathcal{M}(X_i, T_i)$ $(i = 1, 2)$ satisfy $\pi_1(\mu_1) = \pi_2(\mu_2) = \nu \in \mathcal{M}(Y, S)$. Let $\mu_i = \int_Y \mu_{i,y} d\nu(y)$ be the disintegration of $\mu_i$ over $\nu$ for $i = 1, 2$. Set

$$\mu_1 \times_\nu \mu_2 = \int_Y \mu_{1,y} \times \mu_{2,y} d\nu(y);$$

then $\mu_1 \times_\nu \mu_2 \in \mathcal{M}(X_1 \times_Y X_2, T_1 \times_Y T_2)$, and for simplicity write

$$E_2^{\mu_1 \times_\nu \mu_2}(\pi_1 \times_Y \pi_2) = E_2^{\mu_1 \times_\nu \mu_2}(X_1 \times_Y X_2, T_1 \times_Y T_2 | \pi_1 \times_Y \pi_2),$$
$$E_2(\pi_1 \times_Y \pi_2) = E_2(X_1 \times_Y X_2, T_1 \times_Y T_2 | \pi_1 \times_Y \pi_2).$$

The above definitions can be generalized naturally to the case of given $n$ factor maps.

In the remainder of the section, for any given two TDSs $(X_1, T_1)$ and $(X_2, T_2)$, we shall often identify the two product spaces $(X_1 \times X_2) \times (X_1 \times X_2)$ and $(X_1 \times X_1) \times (X_2 \times X_2)$ via the canonical isomorphism: $((x_1, x_2), (x_1', x_2')) \mapsto ((x_1, x_1'), (x_2, x_2'))$. The identification for given $n$-TDSs $(n \geq 2)$ works similarly.

Lemma 6.6: *The following statements hold.*

**1:** *Let $\pi_i\colon (X_i, T_i) \to (Y, S)$ $(i = 1, 2)$ be two factor maps between TDSs and $\mu_i \in \mathcal{M}(X_i, T_i)$ satisfy $\pi_1(\mu_1) = \pi_2(\mu_2) = \nu \in \mathcal{M}(Y, S)$. Then*

$$E_2^{\mu_1 \times_\nu \mu_2}(\pi_1 \times_Y \pi_2) = (E_2^{\mu_1}(\pi_1) \times_Y E_2^{\mu_2}(\pi_2)) \cup (E_2^{\mu_1}(\pi_1) \times_Y s_{\mu_2}^2) \cup (s_{\mu_1}^2 \times_Y E_2^{\mu_2}(\pi_2)).$$

**2:** *Let $\pi_i\colon (X_i, T_i) \to (Y, S)$ $(i = 1, 2)$ be two factor maps between TDSs. Then*

$$E_2(\pi_1 \times_Y \pi_2) = (E_2(\pi_1) \times_Y E_2(\pi_2)) \cup (E_2(\pi_1) \times_Y s_{X_2}^2) \cup (s_{X_1}^2 \times_Y E_2(\pi_2)).$$

*Proof:* Part 1 follows from the proof of Lemma 2 in [PS]. Now we aim to prove part 2.

Take $\mu_i' \in \mathcal{M}(X_i, T_i)$ $(i = 1, 2)$ such that $E_2^{\mu_i'}(\pi_i) = E_2(\pi_i)$. Without loss of generality assume $\mathrm{supp}(\mu_i') = \mathrm{supp}(X_i, T)$ $(i = 1, 2)$ and so $s_{\mu_i'}^2 = s_{X_i}^2$.

Let $\nu_i = \pi_i(\mu_i')$, then $\nu_i \in \mathcal{M}(Y, S)$ $(i = 1, 2)$. It is well-known that there exists $\mu_i^* \in \mathcal{M}(X_i, T_i), i = 1, 2$ such that $\pi_1(\mu_1^*) = \nu_2$ and $\pi_2(\mu_2^*) = \nu_1$. Set $\mu_i = \frac{1}{2}(\mu_i' + \mu_i^*), i = 1, 2$ and $\nu = \frac{1}{2}(\nu_1 + \nu_2)$. Then $\mu_i \in \mathcal{M}(X_i, T_i)$, $\nu \in \mathcal{M}(Y, S)$, and $\pi_i(\mu_i) = \nu$ $(i = 1, 2)$. Obviously, $E_2^{\mu_i}(\pi_i) \supseteq E_2^{\mu_i'}(\pi_i)$ and $s^2_{\mu_i} \supseteq s^2_{\mu_i'}$ for $i = 1, 2$. Hence $E_2^{\mu_i}(\pi_i) = E_2(\pi_i)$ and $s^2_{\mu_i} = s^2_{X_i}$ for $i = 1, 2$. Thus

$$\begin{aligned} &E_2(\pi_1 \times_Y \pi_2) \\ &\subseteq (E_2(\pi_1) \times_Y E_2(\pi_2)) \cup (E_2(\pi_1) \times_Y s^2_{X_2}) \\ &\quad \cup (s^2_{X_1} \times_Y E_2(\pi_2)) \quad \text{(by Proposition 2.3)} \\ &= (E_2^{\mu_1}(\pi_1) \times_Y E_2^{\mu_2}(\pi_2)) \cup (E_2^{\mu_1}(\pi_1) \times_Y s^2_{\mu_2}) \cup (s^2_{\mu_1} \times_Y E_2^{\mu_2}(\pi_2)) \\ &= E_2^{\mu_1 \times_\nu \mu_2}(\pi_1 \times_Y \pi_2) \text{ (by part 1)} \subseteq E_2(\pi_1 \times_Y \pi_2). \end{aligned}$$

That is, $E_2(\pi_1 \times_Y \pi_2) = (E_2(\pi_1) \times_Y E_2(\pi_2)) \cup (E_2(\pi_1) \times_Y s^2_{X_2}) \cup (s^2_{X_1} \times_Y E_2(\pi_2))$. ∎

Denote $X_1 \times_Y \cdots \times_Y X_n$ (resp. $T_1 \times_Y \cdots \times_Y T_n$) by $\prod_{i=1}^n (X_i)_Y$ (resp. $\prod_{i=1}^n (T_i)_Y$). Now we are ready to prove the main result of the section.

THEOREM 6.7: *Let $\pi_i$: $(X_i, T_i) \to (Y, S)$ have rel.-u.p.e. and* $\mathrm{supp}(Y, S) = Y$, $1 \le i \le n$. *Then $(\prod_{i=1}^n (X_i)_Y, \prod_{i=1}^n (T_i)_Y)$ also has rel.-u.p.e. with respect to $(Y, S)$. If in addition all $\pi_i$ are open, then $(\prod_{i=1}^n (X_i)_Y, \prod_{i=1}^n (T_i)_Y)$ is transitive iff so is $Y$.*

*Consequently, if $\pi$: $(X, T) \to (Y, S)$ is open and has rel.-u.p.e., then $\pi$ is weakly mixing iff $\pi$ is weakly mixing of all orders iff $(Y, S)$ is an E-system (applying Theorem 5.4).*

*Proof:* It is sufficient to consider the case when $n = 2$ for the first part of the theorem. Since $(X_i, T_i)$ has rel.-u.p.e. and $\mathrm{supp}(Y, S) = Y$, it follows that $\mathrm{supp}(X_i, T_i) = X_i$, $i = 1, 2$ (by Theorem 5.4). Then $E_2(\pi_i) = R_{\pi_i} \setminus \Delta_2(X_i)$ and $s^2_{X_i} = \Delta_2(X_i)$, $i = 1, 2$. So

$$\begin{aligned} &R_{\pi_1 \times_Y \pi_2} \setminus \Delta_{X_1 \times_Y X_2} = (R_{\pi_1} \times_Y R_{\pi_2}) \setminus \Delta_{X_1 \times_Y X_2} \\ &= ((R_{\pi_1} \setminus \Delta_{X_1}) \times_Y (R_{\pi_2} \setminus \Delta_{X_2})) \cup ((R_{\pi_1} \setminus \Delta_{X_1}) \times_Y \Delta_{X_2}) \cup (\Delta_{X_1} \times_Y (R_{\pi_2} \setminus \Delta_{X_2})) \\ &= (E_2(\pi_1) \times_Y E_2(\pi_2)) \cup (E_2(\pi_1) \times_Y s^2_{X_2}) \cup (s^2_{X_1} \times_Y E_2(\pi_2)). \end{aligned}$$

Thus $R_{\pi_1 \times_Y \pi_2} \setminus \Delta_{X_1 \times_Y X_2} = E_2(\pi_1 \times_Y \pi_2)$ (by Lemma 6.6 (2)), i.e. $\pi_1 \times_Y \pi_2$ has rel.-u.p.e.

When $\pi_1$ and $\pi_2$ are open, $\pi_1 \times_Y \pi_2$ is also open. Since $\pi_1 \times_Y \pi_2$ has rel.-u.p.e., and so it is a u.p.e. extension. If $(Y, S)$ is transitive, then $(X_1 \times_Y X_2, T_1 \times_Y T_2)$ is transitive following from Lemma 6.2. ∎

## 7. Products of relative u.p.e. and c.p.e. extensions

In this final section we shall show that the finite product of rel.-u.p.e. extensions (resp. rel.-c.p.e. extensions) has rel.-u.p.e. (resp. rel.-c.p.e.) iff all factor systems have invariant measures with full support. We start with the discussion on the product of u.p.e. extensions defined by Glasner and Weiss. The following lemma will be used.

LEMMA 7.1: *Let $(X_i, T_i)$ $(i = 1, 2)$ be two TDSs. Then*

$$E_2(X_1 \times X_2) = (E_2(X_1) \times E_2(X_2)) \cup (E_2(X_1) \times s^2_{X_2}) \cup (s^2_{X_1} \times E_2(X_2)).$$

*Proof:* The lemma was obtained by Glasner (see Theorem 3 of [G1] or Theorem 19.24 of [G2] for more details). ∎

Let $\pi_i$: $(X_i, T_i) \to (Y_i, S_i)$ $(1 \le i \le n)$ be $n$ factor maps between TDSs. Define

$$\prod_{i=1}^n \pi_i \colon \prod_{i=1}^n X_i \to \prod_{i=1}^n Y_i$$

such that $\prod_{i=1}^n \pi_i(x_1, \ldots, x_n) = (\pi_1(x_1), \ldots, \pi_n(x_n))$ for each $(x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$; then $\prod_{i=1}^n \pi_i : (\prod_{i=1}^n X_i, \prod_{i=1}^n T_i) \to (\prod_{i=1}^n Y_i, \prod_{i=1}^n S_i)$ is a factor map between TDSs.

Now we show the property of u.p.e. extension is preserved by finite product.

THEOREM 7.2: *Let $\pi_i$: $(X_i, T_i) \to (Y_i, S_i)$ $(1 \le i \le n)$ be $n$ given u.p.e. extensions between TDSs. If* $\operatorname{supp}(Y_i, S_i) = Y_i$ $(1 \le i \le n)$*, then $\prod_{i=1}^n \pi_i$ is a u.p.e. extension.*

*Proof:* It is enough to show the case when $n = 2$. Since $\pi_i$: $(X_i, T_i) \to (Y_i, S_i)$ $(i = 1, 2)$ are u.p.e. extensions, $R_{\pi_1} \setminus \Delta_2(X_1) \subseteq E_2(X_1)$ and $R_{\pi_2} \setminus \Delta_2(X_2) \subseteq E_2(X_2)$.

Let $((x_1, x_2), (x_1', x_2')) \in R_{\pi_1 \times \pi_2} \setminus \Delta_2(X_1 \times X_2)$. Without loss of generality assume $x_1 \ne x_1'$. As any u.p.e. extension is an entropy-generated extension, by Theorem 5.7 one has $\operatorname{supp}(X_2, T_2) = X_2$. Then $s^2_{X_2} = \Delta_2(X_2)$, and so by Lemma 7.1

$$\begin{aligned}
&((x_1, x_2), (x_1', x_2')) \in ((R_{\pi_1} \setminus \Delta_2(X_1)) \times R_{\pi_2} \\
&= (R_{\pi_1} \setminus \Delta_2(X_1)) \times (R_{\pi_2} \setminus \Delta_2(X_2))) \cup (R_{\pi_1} \setminus \Delta_2(X_1)) \times \Delta_2(X_2)) \\
&\subseteq (E_2(X_1) \times E_2(X_2)) \cup (E_2(X_1) \times s^2_{X_2}) \subseteq E_2(X_1 \times X_2).
\end{aligned}$$

That is, $R_{\pi_1 \times \pi_2} \setminus \Delta_2(X_1 \times X_2) \subseteq E_2(X_1 \times X_2)$, i.e. $\pi_1 \times \pi_2$ is a u.p.e. extension. ∎

If $(\xi_i : i \in I)$ is a countable family of finite partitions of $X$, then $\xi = \bigvee_{i\in I} \xi_i$ is called a *measurable partition.* The collection of all sets $B \in \mathcal{B}(X)$, which is the union of some atoms of $\xi$, forms a sub-$\sigma$-algebra of $\mathcal{B}(X)$. By [Rh], every sub-$\sigma$-algebra of $\mathcal{B}(X)$ coincides with a $\sigma$-algebra constructed in this way outside a set of $\mu$-measure zero. Thus sometimes we shall denote them just by the same symbol. Particularly, we can associate each sub-$\sigma$-algebra $\mathcal{G}$ to a measurable partition of $X$ (also denoted by $\mathcal{G}$, but in general we can't ensure $\mathcal{G} \in \mathcal{P}_X$). For any given measurable partition $\xi$ of $(X, \mathcal{B}(X), \mu, T)$, put $\xi^- = \bigvee_{i=1}^{+\infty} T^{-i}\xi$ and $\xi^T = \bigvee_{i=-\infty}^{+\infty} T^{-i}\xi$. We say $\xi$ is a *generating partition* if $\xi^T$ is equal to $\mathcal{B}(X)$ outside a set of $\mu$-measure zero.

To study the product of rel.-u.p.e. and rel.-c.p.e. extensions defined in our paper we need a result similar to Lemma 7.1, whose proof depends on the following result. Let $\pi\colon (X,T) \to (Y,S)$ be a factor map between TDSs and $\mu \in \mathcal{M}(X,T)$. Denote by $P_\mu(\pi)$ the relative Pinsker $\sigma$-algebra $P_\mu(\pi^{-1}\mathcal{B}(Y))$.

LEMMA 7.3: *Let $\pi_i\colon (X_i,T_i) \to (Y_i,S_i)$ be two factor maps between TDSs and $\mu_i \in \mathcal{M}(X_i,T_i)$ $(i=1,2)$. Then*

$$P_{\mu_1\times\mu_2}(\pi_1 \times \pi_2) = P_{\mu_1}(\pi_1) \times P_{\mu_2}(\pi_2).$$

*Proof:* On the one hand, it is obvious that $P_{\mu_1}(\pi_1) \times \{\emptyset, X_2\}, \{\emptyset, X_1\} \times P_{\mu_2}(\pi_2) \subseteq P_{\mu_1\times\mu_2}(\pi_1\times\pi_2)$, and so $P_{\mu_1}(\pi_1)\times P_{\mu_2}(\pi_2) \subseteq P_{\mu_1\times\mu_2}(\pi_1\times\pi_2)$.

On the other hand, by Lemma 3.7 of [Z], $(X_i, \mathcal{B}(X_i), T_i, \mu_i)$ admits a generating partition $P_i$ with $P_i \supseteq \pi_i^{-1}\mathcal{B}(Y_i)$ and $P_{\mu_i}(\pi_i) = \bigcap_{n=0}^{+\infty} T_i^{-n}P_i^-$ $(i=1,2)$. Note that $P_1^- \times P_2^- \supseteq (\pi_1\times\pi_2)^{-1}\mathcal{B}(Y_1\times Y_2)$ is a generating partition of $(X_1\times X_2, \mathcal{B}(X_1\times X_2), \mu_1\times\ \mu_2, T_1\times T_2)$. Then we have (using Lemma 3.6 of [Z])

$$\begin{aligned} P_{\mu_1\times\mu_2}(\pi_1\times\pi_2) &\subseteq \bigcap_{n\geq 0} (T_1\times T_2)^{-n}(P_1^-\times P_2^-) \\ &\subseteq \bigcap_{n\geq 0} (T_1^{-n}P_1^- \times \mathcal{B}(X_2)) = P_{\mu_1}(\pi_1)\times\mathcal{B}(X_2). \end{aligned}$$

Similarly, we have $P_{\mu_1\times\mu_2}(\pi_1\times\pi_2) \subseteq \mathcal{B}(X_1)\times P_{\mu_2}(\pi_2)$. Thus

$$P_{\mu_1\times\mu_2}(\pi_1\times\pi_2) \subseteq (P_{\mu_1}(\pi_1)\times\mathcal{B}(X_2)) \cap (\mathcal{B}(X_1)\times P_{\mu_2}(\pi_2)) = P_{\mu_1}(\pi_1)\times P_{\mu_2}(\pi_2).$$

This means $P_{\mu_1\times\mu_2}(\pi_1\times\pi_2) = P_{\mu_1}(\pi_1)\times P_{\mu_2}(\pi_2)$. ∎

THEOREM 7.4: *Let $\pi_i\colon (X_i,T_i) \to (Y_i,S_i)$ $(i=1,2)$ be two factor maps between TDSs. Then we have*

**1:** *Suppose* $\mu_i \in \mathcal{M}(X_i, T_i)$ $(i = 1, 2)$. *Then*

$$E_2^{\mu_1\times\mu_2}(\pi_1\times\pi_2) = (E_2^{\mu_1}(\pi_1)\times E_2^{\mu_2}(\pi_2))\cup(E_2^{\mu_1}(\pi_1)\times s_{\mu_2}^2)\cup(s_{\mu_1}^2\times E_2^{\mu_2}(\pi_2)).$$

**2:** $E_2(\pi_1 \times \pi_2) = (E_2(\pi_1) \times E_2(\pi_2)) \cup (E_2(\pi_1) \times s_{X_2}^2) \cup (s_{X_1}^2 \times E_2(\pi_2))$.

*Proof:* 1. Assume that $\mu_i = \int_{Z_i} (\mu_i)_{z_i} d\nu_i(z_i)$ is the disintegration of $\mu_i$ over the measure-theoretical relative Pinsker factor $(Z_i, \nu_i)$ of $\pi_i$ $(i = 1, 2)$. Then

$$\lambda_2^{\pi_i}(\mu_i) = \int_{Z_i} (\mu_i)_{z_i} \times (\mu_i)_{z_i} d\nu_i(z_i).$$

By Lemma 7.3, it is clear that $\mu_1 \times \mu_2 = \int_{Z_1\times Z_2} (\mu_1)_{z_1} \times (\mu_2)_{z_2} d\nu_1(z_1)d\nu_2(z_2)$ is the disintegration of $\mu_1 \times \mu_2$ over the measure-theoretical relative Pinsker factor $(Z_1 \times Z_2, \nu_1 \times \nu_2)$ of $\pi_1 \times \pi_2$. Via the canonical isomorphism introduced in section 6 we have

$$\lambda_2^{\pi_1\times\pi_2}(\mu_1 \times \mu_2) = \int_{Z_1\times Z_2} ((\mu_1)_{z_1} \times (\mu_2)_{z_2}) \times ((\mu_1)_{z_1} \times (\mu_2)_{z_2}) d\nu_1(z_1)d\nu_2(z_2).$$

This implies $\lambda_2^{\pi_1\times\pi_2}(\mu_1 \times \mu_2) = \lambda_2^{\pi_1}(\mu_1) \times \lambda_2^{\pi_2}(\mu_2)$, whence

$$\mathrm{supp}(\lambda_2^{\pi_1\times\pi_2}(\mu_1 \times \mu_2)) = \mathrm{supp}(\lambda_2^{\pi_1}(\mu_1)) \times \mathrm{supp}(\lambda_2^{\pi_2}(\mu_2)).$$

Moreover,

$$\begin{aligned}
&E_2^{\mu_1\times\mu_2}(\pi_1 \times \pi_2)\\
&= \mathrm{supp}(\lambda_2^{\pi_1\times\pi_2}(\mu_1 \times \mu_2)) \setminus \Delta_2(X_1 \times X_2) \quad \text{(by Corollary 3.6)}\\
&= (\mathrm{supp}(\lambda_2^{\pi_1}(\mu_1)) \times \mathrm{supp}(\lambda_2^{\pi_2}(\mu_2))) \setminus (\Delta_2(X_1) \times \Delta_2(X_2))\\
&= (\mathrm{supp}(\lambda_2^{\pi_1}(\mu_1)) \setminus \Delta_2(X_1) \times \mathrm{supp}(\lambda_2^{\pi_2}(\mu_2)) \setminus \Delta_2(X_2))\\
&\quad \cup (\mathrm{supp}(\lambda_2^{\pi_1}(\mu_1)) \setminus \Delta_2(X_1) \times s_{\mu_2}^2) \cup (s_{\mu_1}^2 \times \mathrm{supp}(\lambda_2^{\pi_2}(\mu_2)) \setminus \Delta_2(X_2))\\
&= (E_2^{\mu_1}(\pi_1)\times E_2^{\mu_2}(\pi_2)) \cup (E_2^{\mu_1}(\pi_1)\times s_{\mu_2}^2) \cup (s_{\mu_1}^2 \times E_2^{\mu_2}(\pi_2)) \text{ (by Corollary 3.6).}
\end{aligned}$$

2. Take $\mu_i \in \mathcal{M}(X_i, T_i)$ such that $E_2^{\mu_i}(\pi_i) = E_2(\pi_i)$ and $s_{\mu_i}^2 = s_{X_i}^2$ $(i = 1, 2)$. Then

$$\begin{aligned}
&E_2(\pi_1 \times \pi_2) \supseteq E_2^{\mu_1\times\mu_2}(\pi_1 \times \pi_2)\\
&= (E_2^{\mu_1}(\pi_1) \times E_2^{\mu_2}(\pi_2)) \cup (E_2^{\mu_1}(\pi_1) \times s_{\mu_2}^2) \cup (s_{\mu_1}^2 \times E_2^{\mu_2}(\pi_2)) \quad \text{(by part 1)}\\
&= (E_2(\pi_1) \times E_2(\pi_2)) \cup (E_2(\pi_1) \times s_{X_2}^2) \cup (s_{X_1}^2 \times E_2(\pi_2))\\
&\supseteq E_2(\pi_1 \times \pi_2) \quad \text{(by Proposition 2.3).}
\end{aligned}$$

This means $E_2(\pi_1\times\pi_2)=(E_2(\pi_1)\times E_2(\pi_2))\cup(E_2(\pi_1)\times s_{X_2}^2)\cup(s_{X_1}^2\times E_2(\pi_2))$. ∎

Let $(X, T)$ be a TDS and $\emptyset \neq R \subseteq X \times X$. Denote by $\langle R\rangle$ the *icer* generated by $R$. The following lemma is well-known (for example, Lemma 1 of [PS]).

Lemma 7.5: *Let* $(X_i, T_i)$ $(i = 1, 2)$ *be two TDSs and* $\Delta_2(X_i) \subseteq A_i \subseteq X_i \times X_i$. *Then* $\langle A_1 \times A_2 \rangle = \langle A_1 \rangle \times \langle A_2 \rangle$.

With the above preparations, we have

Theorem 7.6: *Let* $\pi_i\colon (X_i, T_i) \to (Y_i, S_i)$ *be two factor maps between TDSs.*

**1:** *If* $\pi_1$ *and* $\pi_2$ *are both rel.-c.p.e., then* $\pi_1 \times \pi_2$ *is also rel.-c.p.e. iff* $\mathrm{supp}(Y_i, S_i) = Y_i$ $(i = 1, 2)$.

**2:** *If* $\pi_1$ *and* $\pi_2$ *are both rel.-u.p.e., then* $\pi_1 \times \pi_2$ *is also rel.-u.p.e. iff* $\mathrm{supp}(Y_i, S_i) = Y_i$ $(i = 1, 2)$.

*Proof:* We shall give a proof of part 1. Part 2 follows from similar discussions.

Let $R$ (resp. $R_1$, $R_2$) be the icer generated by $E_2(\pi_1 \times \pi_2) \cup \Delta_2(X_1 \times X_2)$ (resp. $E_2(\pi_1) \cup \Delta_2(X_1)$, $E_2(\pi_2) \cup \Delta_2(X_2)$). For $i = 1, 2$, set

$$W_i \doteq \{(x_1, x_2) \in X_i \times X_i : \pi_i(x_1) = \pi_i(x_2) \in \mathrm{supp}(Y_i, S_i)\}.$$

As $\pi_1$ and $\pi_2$ are both rel.-c.p.e., by Proposition 5.3 and Theorem 5.4 one has $R_i = R_{\pi_i}$ and $W_i \setminus \Delta_2(X_i) = R_{\pi_i} \setminus \Delta_2(X_i) \neq \emptyset$, $i = 1, 2$.

First assume that $\mathrm{supp}(Y_i, S_i) = Y_i$ $(i = 1, 2)$. By Theorem 5.4, $\mathrm{supp}(X_i, T_i) = X_i$ $(i = 1, 2)$. Using Theorem 7.4 we obtain

$$E_2(\pi_1 \times \pi_2) \cup \Delta_2(X_1 \times X_2) = (E_2(\pi_1) \cup \Delta_2(X_1)) \times (E_2(\pi_2) \cup \Delta_2(X_2)),$$

which implies that $R = R_1 \times R_2 = R_{\pi_1 \times \pi_2}$ (by Lemma 7.5), i.e. $\pi_1 \times \pi_2$ is rel.-c.p.e. (by Proposition 5.3).

Now assume that $\pi_1 \times \pi_2$ is rel.-c.p.e. and so $R = R_{\pi_1 \times \pi_2}$ (by Proposition 5.3). Set

$$W \doteq \Delta_2(X_1 \times X_2) \cup \prod_{i=1}^{2} W_i \subseteq R_{\pi_1 \times \pi_2}.$$

It is not hard to obtain that $W$ is an icer containing $E_2(\pi_1 \times \pi_2) \cup \Delta_2(X_1 \times X_2)$. Then $W = R_{\pi_1 \times \pi_2}$, and so

$$(W_1 \setminus \Delta_2(X_1)) \times \{(x, x) \in X_2 \times X_2 : \pi_2(x) \notin \mathrm{supp}(Y_2, S_2)\} \subseteq R_{\pi_1 \times \pi_2} \setminus W = \emptyset,$$

which implies $\mathrm{supp}(Y_2, S_2) = Y_2$, as $W_1 \setminus \Delta_2(X_1) \neq \emptyset$. Similarly, $\mathrm{supp}(Y_1, S_1) = Y_1$. This finishes the proof. ∎

## 8. Appendix

In the appendix, following the ideas in [BHR] we will discuss the relationship between r.t.-entropy pairs and asymptotic pairs. We shall only give a brief description; the reader can refer to [BHR] for more details.

Let $(X,T)$ be a TDS, $\mu \in \mathcal{M}(X,T)$ and $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{B}(X)$. Define the *conditional product* $\mu \times_{\mathcal{G}} \mu$ of $\mu$ relative to $\mathcal{G}$ to be the measure on $(X^{(2)}, \mathcal{B}(X) \times \mathcal{B}(X))$ determined completely by

$$\text{for all } A, B \in \mathcal{B}(X), \quad \mu \times_{\mathcal{G}} \mu(A \times B) = \int_X \mathbb{E}(1_A|\mathcal{G})(x)\mathbb{E}(1_B|\mathcal{G})(x)d\mu(x).$$

Clearly, $\mu \times_{\mathcal{G}} \mu$ is a probability measure, its two projections on $X$ are both equal to $\mu$, and $\mathrm{supp}(\mu \times_{\mathcal{G}} \mu) \supseteq \{(x,x) \in X^{(2)} : x \in \mathrm{supp}(\mu)\}$. Since $\mu$ is $T$-invariant, and

$$\mathbb{E}(1_A|\mathcal{G})(Tx) = \mathbb{E}(1_{T^{-1}A}|T^{-1}\mathcal{G})(x) \quad \text{for } \mu\text{-a.e. } x \in X,$$

we have $\mu \times_{T^{-1}\mathcal{G}} \mu = (T \times T)^{-1}(\mu \times_{\mathcal{G}} \mu)$. Moreover, the measure $\mu \times_{\mathcal{G}} \mu$ is $T \times T$-invariant if $\mathcal{G}$ is $T$-invariant (i.e. $\mathcal{G} = T^{-1}\mathcal{G}$). In fact, by standard arguments, for each pair of bounded Borel functions $f, g$ on $X$ one has

$$\int_{X \times X} f(x)g(y)d(\mu \times_{\mathcal{G}} \mu)(x,y) = \int_X \mathbb{E}(f|\mathcal{G})(x)\mathbb{E}(g|\mathcal{G})(x)d\mu(x).$$

For any sub-$\sigma$-algebra $\mathcal{G}'$ of $\mathcal{B}(X)$, we set

$$\Delta_{\mathcal{G}'} = \{(x_1,x_2) \in X \times X : x_1 \text{ and } x_2 \text{ belong to one and the same atom of } \mathcal{G}'\}.$$

If $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sub-$\sigma$-algebras of $\mathcal{B}(X)$ with $\bigcap_{n\geq 1}\mathcal{G}_n = \mathcal{G}$, then for all $A, B \in \mathcal{B}(X)$ one has

$$\lim_{n\to\infty} \mu \times_{\mathcal{G}_n} \mu(A \times B) = \mu \times_{\mathcal{G}} \mu(A \times B),$$

and the sequence $(\mu \times_{\mathcal{G}_n} \mu)_{n\in\mathbb{N}}$ converges weakly to $\mu \times_{\mathcal{G}} \mu$ (Lemma 5 of [BHR]). Moreover, in the sense of neglecting a subset of measure zero, $\Delta_{\mathcal{G}} \in \mathcal{B}(X) \times \mathcal{B}(X)$ and $\mu \times_{\mathcal{G}} \mu$ is concentrated on $\Delta_{\mathcal{G}}$ (Lemma 6 of [BHR]).

Now let $\pi\colon (X,T) \to (Y,S)$ be a factor map between TDSs and $\mu \in \mathcal{M}(X,T)$. Lemma 3.7 in [Z] tells us that $(X, \mathcal{B}(X), T, \mu)$ admits a generating partition $P$ with $P \supseteq \pi^{-1}\mathcal{B}(Y)$ and $P_\mu(\pi^{-1}\mathcal{B}(Y)) = \bigcap_{n=0}^{+\infty} T^{-n}P^-$ such that any pair of points belonging to the same atom of $P^-$ is asymptotic. Moreover, if $h_\mu(T|\pi) > 0$ then the $\sigma$-algebra $P^-$ and $\mathcal{B}_\mu$ do not coincide up to sets of $\mu$-measure zero, where $\mathcal{B}_\mu$ is the completion of $\mathcal{B}(X)$ under $\mu$. Thus in the notations of the above discussions, we have $\Delta_{P^-} \subseteq AP(X,T) \cap R_\pi$. Put

$$\mathcal{F}_n = T^{-n}P^- \quad \text{and} \quad \nu_n = \mu \times_{\mathcal{F}_n} \mu \quad \text{for all } n \geq 0.$$

Then we have

$$\Delta_{\mathcal{F}_n} = (T\times T)^{-n}\Delta_{\mathcal{F}} \subseteq AP(X,T)\cap R_\pi \quad \text{and} \quad \nu_n = (T\times T)^{-n}\nu_0.$$

Thus $\nu_n$ is concentrated on $AP(X,T)\cap R_\pi$. Since $\lambda_2^\pi(\mu) = \mu \times_{P_\mu(\pi^{-1}\mathcal{B}(Y))} \mu$ and the decreasing sequence $(\mathcal{F}_n)_{n\geq 0}$ of sub-$\sigma$-algebras of $\mathcal{B}(X)$ satisfies

$$\bigcap_{n\geq 0} \mathcal{F}_n = \bigcap_{n=0}^{+\infty} T^{-n}P^- = P_\mu(\pi^{-1}\mathcal{B}(Y)),$$

we have, as $n\to\infty$, $\nu_n(A\times B)\to \lambda_2^\pi(\mu)(A\times B)$ for all $A,B\in\mathcal{B}(X)$ and the sequence $(\nu_n)_{n\geq 0}$ of measures on $X^{(2)}$ converges weakly to $\lambda_2^\pi(\mu)$.

Lemma 8.1: *Let $\pi$: $(X,T)\to(Y,S)$ be a factor map between TDSs and $\mu\in\mathcal{M}(X,T)$. Suppose that $Q=\{A_1,A_2\}\in\mathcal{P}_X$ satisfies $h_\mu(T,Q|\pi)>0$. Then there exist $x_1\in A_1$ and $x_2\in A_2$ such that $(x_1,x_2)\in AP(X,T)\cap R_\pi$.*

*Proof:* Assume the contrary: $(A_1\times A_2)\cap AP(X,T)\cap R_\pi=\emptyset$. In the notations of the above discussions, since for all $n\geq 0$, $\nu_n$ is concentrated on $AP(X,T)\cap R_\pi$, and so $\nu_n(A_1\times A_2)=0$ for all $n\geq 0$. Then

$$\lambda_2^\pi(\mu)(A_1\times A_2) = \lim_{n\to\infty}\nu_n(A_1\times A_2)=0.$$

By Lemma 3.4, $h_\mu(T,Q|\pi)=0$, a contradiction. ∎

As an application of Lemma 8.1, one has

Proposition 8.2: *Let $\pi_2$: $(X,T)\to(Y,S)$ and $\pi_1$: $(Y,S)\to(Z,\theta)$ be two factor maps between TDSs. Suppose $R_{\pi_2}\supseteq AP(X,T)\cap R_{\pi_1\pi_2}$. Then $h_{\text{top}}(S|\pi_1)=0$.*

*Proof:* Assume the contrary: $h_{\text{top}}(S|\pi_1)>0$. By (3.3) there exists $\nu\in\mathcal{M}(Y,S)$ and $Q=\{A_1,A_2\}\in\mathcal{P}_Y$ such that $h_\nu(S,Q|\pi_1)>0$. Let $\mu\in\mathcal{M}(X,T)$ satisfy $\pi_2\mu=\nu$. Then $h_\mu(T,\pi_2^{-1}(Q)|\pi_1\pi_2)=h_\nu(S,Q|\pi_1)>0$. By Lemma 8.1, there exist $x_i\in\pi_2^{-1}(A_i), i=1,2$ such that $(x_1,x_2)\in AP(X,T)\cap R_{\pi_1\pi_2}$. Since $R_{\pi_2}\supseteq AP(X,T)\cap R_{\pi_1\pi_2}$, $(x_1,x_2)\in R_{\pi_2}$. So $\pi_2(x_1)(=\pi_2(x_2))\in A_1\cap A_2$, a contradiction with $Q\in\mathcal{P}_Y$. ∎

The following theorem interprets the relationship between the set of r.t.-entropy pairs and the set of asymptotic pairs.

THEOREM 8.3: *Let* $\pi\colon (X,T) \to (Y,S)$ *be a factor map between TDSs. Then* $E_2(X,T|\pi) \subseteq \overline{AP(X,T) \cap R_\pi}$.

*Proof:* By Theorem 4.1, there exists $\mu \in \mathcal{M}(X,T)$ such that $E_2^\mu(X,T|\pi) = E_2(X,T|\pi)$. Using the notations of the above discussions to such $\mu$, since $(\nu_n)_{n\geq 0}$ converges weakly to $\lambda_2^\pi(\mu)$, and $\nu_n(\overline{AP(X,T) \cap R_\pi}) = 1$ for all $n \geq 0$, we get $\lambda_2^\pi(\mu)(\overline{AP(X,T) \cap R_\pi}) = 1$ and so $\mathrm{supp}(\lambda_2^\pi(\mu)) \subseteq \overline{AP(X,T) \cap R_\pi}$. Thus $E_2^\mu(X,T|\pi) \subseteq \overline{AP(X,T) \cap R_\pi}$ follows from the fact of $E_2^\mu(X,T|\pi) = \mathrm{supp}(\lambda_2^\pi(\mu)) \setminus \Delta_2(X)$ (Corollary 3.6). Obviously, $R_1 \supseteq R_2$. Ends the proof. ∎

## References


[B1] F. Blanchard, *Fully positive topological entropy and topological mixing,* in *Symbolic Dynamics and its Applications*, Contemporary Mathematics **135** (1992), 95–105.

[B2] F. Blanchard, *A disjointness theorem involving topological entropy,* Bulletin de la Société Mathématique de France **121** (1993), 465–478.

[BGH] F. Blanchard, E. Glasner and B. Host, *A variation on the variational principle and applications to entropy pairs,* Ergodic Theory and Dynamical Systems **17** (1997), 29–43.

[BHR] F. Blanchard, B. Host and S. Ruette, *Asymptotic pairs in positive-entropy systems,* Ergodic Theory and Dynamical Systems **22** (2002), 671–686.

[BL] F. Blanchard and Y. Lacroix, *Zero-entropy factors of topological flows,* Proceedings of the American Mathematical Society **119** (1993), 985–992.

[B-R] F. Blanchard, B. Host, A. Maass, S. Martínez and D. Rudolph, *Entropy pairs for a measure,* Ergodic Theory and Dynamical Systems **15** (1995), 621–632.

[DGS] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spacesw,* Lecture Notes in Mathematics **527**, Springer-Verlag, New York, 1976.

[DS] T. Downarowicz and J. Serafin, *Fiber entropy and conditional variational principles in compact non-metrizable spaces,* Fundamenta Mathematicae **172** (2002), 217–247.

[DYZ] D. Dou, X. Ye and G. H. Zhang, *Entropy sequences and maximal entropy sets,* Nonlinearity **19** (2006), 53–74.

[F] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation,* Mathematical Systems Theory **1** (1967), 1–49.

[G1] E. Glasner, *A simple characterization of the set of μ-entropy pairs and applications,* Israel Journal of Mathematics **102** (1997), 13–27.

[G2] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs 101, American Mathematical Society, Providence, RI, 2003.

[GW1] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions,* Nonlinearity **6** (1993), 1067–1075.

[GW2] E. Glasner and B. Weiss, *Topological entropy of extensions,* in *Ergodic Theory and its Connections with Harmonic Analysis*, Cambridge University Press, 1995, pp. 299–307.

[GW3] E. Glasner and B. Weiss, *Locally equicontinuous dynamical systems,* Colloquium Mathematicum **84–85** (2000), 345–361.

[GW4] E. Glasner and B. Weiss, *On the interplay between measurable and topological dynamics,* in *Handbook of Dynamical Systems*, Vol. 1B (Hasselblatt and Katok, eds.), Elsevier, 2006, pp. 597–648.

[HSY] W. Huang, S. Shao and X. Ye, *Mixing via sequence entropy,* in *Algebraic and topological dynamis*, Contemporary Mathematics **358**, American Mathematical Society, Providence, RI, 2005, pp. 101–122.

[HY1] W. Huang and X. Ye, *Topological complexity, return times and weak disjointness,* Ergodic Theory and Dynamical Systems **24** (2004), 825–846.

[HY2] W. Huang and X. Ye, *A local variational relation and applications,* Israel Journal of Mathematics **151** (2006), 237–279.

[HYZ] W. Huang, X. Ye and G. H. Zhang, *A local variational principles for conditional entropy,* Ergodic Theory and Dynamical Systems **26** (2006), 219–245.

[LS] M. Lemanczyk and A. Siemaszko, *A note on the existence of a largest topological factor with zero entropy,* Proceedings of the American Mathematical Society **129** (2001), 475–482.

[LW] F. Ledrappier and P. Walters, *A relativised variational principle for continuous transformations,* Journal of the London Mathematical Society (2) **16** (1977), 568–576.

[P] W. Parry, *Topics in Ergodic Theory,* Cambridge Tracts in Mathematics 75, Cambridge University Press, Cambridge–New York, 1981.

[PS] K. K. Park and A. Siemaszko, *Relative topological Pinsker factors and entropy pairs,* Monatshefte für Mathematik **134** (2001), 67–79.

[Rh] V. A. Rohlin, *On the fundamental ideas of measure theory,* Matematicheskii Sbornik (N. S.) **25**(67) **1** (1949), 107–150; English Translation: American Mathematical Society Translations (1) **10**(1962), 1–54.

[Rm] P. Romagnoli, *A local variational principle for the topological entropy,* Ergodic Theory and Dynamical Systems **23** (2003), 1601–1610.

[W] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics **79**, Springer-Verlag, New York–Berlin, 1982.

[Z] G. H. Zhang, *Relative entropy, asymptotic pairs and chaos*, Journal of the London Mathematical Society(2) **73** (2006), 157–172.